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Study of the Normality and Continuity for the Mixed Integral Equations with Phase-Lag Term

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Abstract

In this paper, the existence and uniqueness solution of the Fredholm-Volterra integral equations (F-VIEs) are considered in the space $L_2[0,1] \times C^n[0,T],\ 0 \leq T < 1$. Using a numerical technique, F-VIEs lead to a system of linear Fredholm integral equations (SLFIEs). Also, the normality and the continuity of integral operator are discussed. The Trapezoidal Rule is used to get the solution of SLFIEs. Finally, numerical results are discussed and the error estimate is computed.

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Keywords: Fredholm-Volterra integral equations, Mixed integral equation, Linear system of Fredholm integral equations, Continuous kernel, Trapezoidal Rule, Phase-Lag

1 Introduction

Many problems of mathematical physics, contact problems in the theory of elasticity and mixed problems of mechanics of continuous media are reduced to mixed type of integral equations, see [1,11]. For this many different methods are used to solve the integral equations analytically, see [6–8]. In addition, for numerical methods, we refer to [5].

Phase-Lag has a very important role in our applied life and there are currently One, Dual and Three-Phases and each phase has a different applications. For example, the Three-Phase-Lag model incorporates the microstructural interaction effect in the fast-transient process of heat transport, see [4].

In this paper, we consider the Fredholm-Volterra integral equations of the second kind with continuous kernels with respect to position and time. The existence and uniqueness of the solution, under certain conditions, are discussed in the space $L_2[0,1]\times C^n[0,T],\ 0\le T<1$. We use a numerical method to transform the Fredholm-Volterra integral equations to a linear system of Fredholm integral equations [2,9].

Consider the linear mixed integral equation,

$$y(u,t+q)=g(u,t)+\lambda\int_0^1 k(u,v)y(v,t)dv+\lambda\int_0^t \Phi(t,\tau)y(u,\tau)d\tau,\quad (q<<1),\tag{1}$$

where q is the Phase-Lag is positive, very small and assumed to be intrinsic properties of the medium. The constant parameter λ may be complex and has many physical meanings, the function $y(u,t)$ is unknown in the Banach space and continuous with their derivative with respect to time in the space $L_2[0,1]\times C^n[0,T],\ 0\le T<1$, where $[0,1]$ is the domain of integration with respect to the position and the time $t\in[0,T]$ and it's called the potential function of the mixed integral equation. The kernel $\Phi(t,\tau)$ is positive and continuous in $C^n[0,T]$ and the known function $g(u,t)$ is continuous and its derivatives with respect to position and time, while the kernel of position $k(u,v)$ is a continuous function.
Using Taylor Expansion after neglecting the second derivative in the equation (1) we get,

$$y(u,t)+q\frac{\partial y(u,t)}{\partial t}=g(u,t)+\lambda\int_0^1 k(u,v)y(v,t)dv+\lambda\int_0^t \Phi(t,\tau)y(u,\tau)d\tau,\quad (q<<1),\tag{2}$$

with initial condition,

$$y(u,0)=f(u).\tag{3}$$

The equation (2) with initial condition (3) is called Integro-Differential Equation for the Phase-Lag. The Integro-Differential Equation is a kind of functional equation that has associate integral and derivatives of unknown

function. These equations were named after the leading mathematicians who have first studied them such as Fredholm, Volterra. Fredholm and Volterra equations are the most encountered types. There is, formally only one difference between them, in the Fredholm equation the region of integration is fixed where in the Volterra equation the region is variable. Integro-Differential Equations (IDEs) are given as a combination of differential and integral equations. In the recent study there is a growing interest to solve Integro-Differential Equations [10, 12].
Integrating Eq.(2) and using initial condition (3) we get,

$$\begin{aligned} q\ y(u,t) = qf(u) + \int_0^t g(u,z)dz - \int_0^t y(u,z)dz + \lambda \int_0^t \int_0^1 k(u,v)y(v,z)dvdz \\ + \lambda \int_0^t \int_0^z \Phi(z,\tau)y(u,\tau)d\tau dz, \qquad (q << 1). \end{aligned} \tag{4}$$

Interchanging the order of integration over the triangular domain in the τz-plane reveals that the equation (4) becomes,

$$\begin{aligned} q\ y(u,t) = qf(u) + \int_0^t g(u,\tau)d\tau + \lambda \int_0^t \int_0^1 k(u,v)y(v,\tau)dvd\tau \\ + \int_0^t [\lambda\Psi(t,\tau) - 1]y(u,\tau)d\tau, \qquad (q << 1). \end{aligned} \tag{5}$$

Where,

$$\Psi(t,\tau) = \int_\tau^t \Phi(z,\tau)dz.$$

The equation (5) is called Mixed Integral Equation with Phase-Lag Term in position and time.

2 The existence and uniqueness of solution of the F-VIEs

In order to guarantee the existence of a unique solution of equation (5), we assume through this work the following conditions:

(i) The kernel $k(u,v) \in L_2([0,1]\times[0,1])$, $u,v \in [0,1]$ satisfies $|k(u,v)| < E$, E is a constant.

(ii) The continuous function $\Psi(t,\tau) \in C^n([0,T])$ and satisfies $|\Psi(t,\tau)| \leq B$, $s.t$ B is a constant, $\forall\, t,\tau \in [0,T]$.

(iii) The function $f(u)$ is continuous and satisfies the condition $|f(u)| \leq D,\ s.t\quad D$ is a constant.

(iv) The function $g(u,\tau)$ with its partial derivatives with respect to the position and time are continuous in the space $L_2[0,1]\times C^n[0,T],\ 0\leq T<1$ and its norm is defined as,

$$\|g(u,\tau)\| = \sum_{k=0}^{n} \max_{0<\tau\leq T} \int_0^{\tau} ([\int_0^1 g^2(u,z)du]^{\frac{1}{2}})^{(k)}dz = Q, \quad Q \text{ is a constant.}$$

3 The normality and continuity of the integral operator

To prove the existence and the uniqueness solution of equation (5), we use the normality and continuity of the integral operator, with the help of Banach fixed point. For this the integral equation (5) can be written in the integral operator form,

$$\overline{V}y = f(u) + \frac{1}{q}\int_0^t g(u,\tau)d\tau + Vy(u,t), \tag{6}$$

and,

$$Vy = Ky + [\Psi - I]y, \tag{7}$$

where,

$$Ky = \frac{\lambda}{q}\int_0^t\int_0^1 k(u,v)y(v,\tau)dvd\tau,$$
$$[\Psi - I]y = \frac{1}{q}\int_0^t [\lambda\Psi(t,\tau)-1]y(u,\tau)d\tau.$$

Theorem 3.1. *If the conditions (i)-(iv) are satisfied and the integral operator (7) is a normal and continuous, then equation (5) has an unique solution $y(u,t)$ in the Banach space $L_2[0,1]\times C^n[0,T],\ 0\leq T<1$, under the condition,*

$$|\,\lambda\,| < \frac{|q|+T+1}{(T+1)[2E+B]}; \qquad (q\neq 0). \tag{8}$$

4 The reduced system of Fredholm integral equations and its solution

4.1 Quadratic numerical method

The importance of Quadratic numerical method comes from its wide applications in mathematical physics problems, wherever the eigenvalues and eigenfunctions of the integral equations are often studied and discussed. Also, this method has wide applications within the applied sciences especially within the theory of elasticity, mixed problems of mechanics area and contact problem.

In this subsection, we tend to use this numerical technique to reduce the F-VIEs to linear SFIEs of the second kind. We divide the interval $[0,T],\ 0 \leq T < 1$, as $0 = t_0 < t_1 < ... < t_i < ... < t_N = T$, where $t = t_i,\ i = 0,1,...,N$, to get

$$\begin{aligned} q\ y(u,t_i) = qf(u) + \int_0^{t_i} g(u,\tau)d\tau + \lambda \int_0^{t_i}\int_0^1 k(u,v)y(v,\tau)dvd\tau \\ + \int_0^{t_i} [\lambda\Psi(t_i,\tau) - 1]y(u,\tau)d\tau, \qquad (q << 1), \end{aligned} \tag{9}$$

using the quadrature formula, for the Volterra integral terms [3], we have

$$\int_0^{t_i}\int_0^1 k(u,v)y(v,\tau)dvd\tau = \sum_{j=0}^{i} \mu_j \int_0^1 k(u,v)y(v,t_j)dv + O(\hbar_i^{\wp+1}), \tag{10}$$

$$\int_0^{t_i} [\Psi(t_i,\tau) - 1]y(u,\tau)d\tau = \sum_{j=0}^{i} \mu_j[\Psi(t_i,t_j) - 1]y(u,t_j) + O(\hbar_i^{\wp+1}), \tag{11}$$

$$\int_0^{t_i} g(u,\tau)d\tau = \sum_{j=0}^{i} \mu_j g(u,t_j) + O(\hbar_i^{\wp+1}), \tag{12}$$

where, $(\hbar_i^{\wp+1} \longrightarrow 0,\ \wp > 0)$ and $\hbar$ denotes the step size of the partition,

$$\hbar_i = \max_{0\leq j\leq i} \rho_j \quad \text{and} \quad \rho_j = t_{j+1} - t_j.$$

The values of the weight formula μ_j and the constant $\wp_1$ are depend on the number of derivatives of $\Psi(t,\tau),\ \forall\tau \in [0,T]$, with respect to t. More information for the characteristic points and the quadrature coefficients are found in [5].
Using of equations (10), (11) and (12) in the equation (9) we get,

$$\begin{aligned} q\ y(u,t_i) = qf(u) + \sum_{j=0}^{i} \mu_j g(u,t_j) + \lambda \sum_{j=0}^{i} \mu_j \int_0^1 k(u,v)y(v,t_j)dv \\ + \sum_{j=0}^{i} \mu_j[\lambda\Psi(t_i,t_j) - 1]y(x,t_j), \end{aligned} \tag{13}$$

using the following notations:

$$y(u,t_i) = y_i(u), \quad g(u,t_j) = g_j(u), \quad \Psi(t_i,t_j) = \Psi_{i,j}.$$

We can write (13) in the following form:

$$\delta_i y_i(u) = F(u) + \lambda \sum_{j=0}^{i} \mu_j \int_0^1 k(u,v) y_j(v) dv. \tag{14}$$

Where,

$$\delta_i = (q - \lambda_i), \ \ \lambda_i = \mu_i(\lambda \Psi_{i,i} - 1), \ \ F(u) = qf(u) + \sum_{j=0}^{i} \mu_j g_j(u) + \sum_{j=0}^{i-1} \mu_j [\lambda \psi_{i,j} - 1] y_j(u).$$

The equation (14) leads us to say that, we have a finite number of unknown functions $y_i(u), \ i = 0, 1, ..., N$ corresponding to the time interval $0 = t_0 < t_1 < ... < t_i < ... < t_N = T$, and the solutions of $y_i(u)$ must be known. For this, we say that the equation (14), for $\delta_i \neq 0$, represents a finite system of Fredholm integral equations of the second kind with continuous kernels with respect to position, while, for $\delta_i = 0$, we have a finite system of Fredholm integral equations of the first kind.

The solution of the system (14), for $\delta_i \neq 0$, can be obtained using different methods. For example, by using the collocation method [8] and Galerkin method [3]. Also, a variation of Nyström method is used in [5] to solve the system (14). To obtain the solution of the system (14), we shall use the trapezoidal rule. If we obtain, firstly, the value of $y_0(u)$, and let $i = 0$ in (14), we get

$$\delta_0 y_0(u) = qf(u) + \mu_0 g_0(u) + \lambda \mu_0 \int_0^1 k(u,v) y_0(v) dv, \qquad \delta_0 = q - \mu_0(\lambda \Psi_{0,0} - 1). \tag{15}$$

After obtaining the solution of equations (15), we can use the mathematical induction to obtain the general solution of (14).

4.2 The trapezoidal rule (TR)

In this section, we can find the solution of the linear algebraic integral system (14) by applying the TR. We will write the term of integration of the equation (14) as follows:

$$\int_0^1 K(u,v) dv = \int_0^1 k(u,v) y_j(v) dv. \tag{16}$$

In this method, we approximate $K(u,v)$ with a collection of surface segments and integrate across each of these. Let P be a partition of $[0,1]$ into n subintervals of equal width, $P : 0 = v_0 < v_1 < ... < v_n = 1$, where $(v_r - v_{r-1}) = \frac{1}{n}$ for $r = 1, 2, ..., n$.

Here, instead of approximating $K(u,v)$ with a horizontal surface segment over $[v_{r-1}, v_r]$, we shall approximate $K(u,v)$ with the surface segment, that has the lines $(v_{r-1}, K(u, v_{r-1}))$ and $(v_r, K(u, v_r))$ as its endpoints-points that lie on the surface of $Z = K(u, v)$. Approximating the surface of $Z = K(u, v)$ with surface segments across successive intervals to obtain the TR. By evaluating the integral on the Eq.(16), we obtain

$$\int_0^1 k(u,v)y_j(v)dv \approx \frac{\Delta v}{2}\left[k(u,v_0)y_j(v_0) + 2\sum_{r=1}^{n-1} k(u,v_r)y_j(v_r) + k(u,v_n)y_j(v_n)\right]. \tag{17}$$

Where,

$$v_0 = 0, \quad v_n = 1, \quad v_r = \frac{r}{n}, \qquad 1 \le r \le n-1. \tag{18}$$

It is known by this name TR because on each subinterval $[v_{r-1}, v_r]$, we are approximating the region bounded by the surface $Z = K(u, v)$, the v-axis, and the lines $v = v_{r-1}$ and $v = v_r$, with a region having a trapezoidal shape.

Theorem 4.1. *Suppose that* $\frac{\partial^2}{\partial v^2} k(u,v)y_j(v)$ *exists on* $[0, 1]$. *Then for a positive integer* n *we have*

$$\int_0^1 k(u,v)y_j(v)dv = T_n + E_n,$$

where,

$$T_n = \frac{\Delta v}{2}\left[k(u,v_0)y_j(v_0) + 2\sum_{r=1}^{n-1} k(u,v_r)y_j(v_r) + k(u,v_n)y_j(v_n)\right],$$

and the error E_n *is given by*

$$E_n = -\frac{1}{12n^2}\frac{\partial^2[k(u,c)y_j(c)]}{\partial v^2},$$

for some value c *in* $[0, 1]$. *Since the number* c *is not specified in this theorem, we are unable to use this to determine the exact value of* E_n *for functions* $k(u,v)y_j(v)$ *in general. However, one of the implications here is that the magnitude of the error has the bounds*

$$\frac{1}{12n^2}\min_{0\le v\le 1}\left|\frac{\partial^2}{\partial v^2}k(u,v)y_j(v)\right| \le |E_n| \le \frac{1}{12n^2}\max_{0\le v\le 1}\left|\frac{\partial^2}{\partial v^2}k(u,v)y_j(v)\right|.$$

Thus if $\frac{\partial^2}{\partial v^2} k(u,v)y_j(v)$ *is never 0 on* $[0, 1]$, *then the error* E_n *must be non-zero.*

4.3 The procedure of solution

By substituting from (17) in (14), we have

$$\delta_i y_i(u) = F(u) + \frac{\lambda \Delta v}{2} \sum_{j=0}^{i} \mu_j [k_0(u) y_{j,0} + k_n(u) y_{j,n} + 2 \sum_{r=1}^{n-1} k_r(u) y_{j,r}], \qquad (19)$$

where $k_0(u) = k(u, v_0)$. Since equation (19) must hold for all values of u, it must hold for values of u equal to those chosen for the quadrature points so that, $u_m = v_r,\ (m = 0, 1, 2, ..., n)$.
By picking those particular points we can generate the following linear system of equations from the equation (19),

$$\delta_i y_i(u_m) = F(u_m) + \frac{\lambda \Delta v}{2} \sum_{j=0}^{i} \mu_j [k_0(u_m) y_{j,0} + k_n(u_m) y_{j,n} + 2 \sum_{r=1}^{n-1} k_r(u_m) y_{j,r}]. \qquad (20)$$

We can write (20) in the following form,

$$\delta_i y_{i,m} = F_m + \frac{\lambda \Delta v}{2} \sum_{j=0}^{i} \mu_j [k_{0,m} y_{j,0} + k_{n,m} y_{j,n} + 2 \sum_{r=1}^{n-1} k_{r,m} y_{j,r}]. \qquad (21)$$

Define the function,

$$H_{i,m}(y_{i,0}, y_{i,1}, ..., y_{i,n}) = \frac{1}{\delta_i} F_m + \frac{\lambda \Delta v}{2\delta_i} \sum_{j=0}^{i} \mu_j [k_{0,m} y_{j,0} + k_{n,m} y_{j,n} + 2 \sum_{r=1}^{n-1} k_{r,m} y_{j,r}], \qquad (22)$$

where, $(m = 0, 1, ..., n,\ j = 0, 1, ..., i,\ i = 0, 1, ..., N)$. Then equation (22) represents a system of linear algebraic equations which may be written as a matrix system,

$$\begin{pmatrix} y_{i,1} \\ y_{i,2} \\ . \\ . \\ . \\ y_{i,n} \end{pmatrix} = \begin{pmatrix} H_{i,1}(y_{i,0}, y_{i,1}, ..., y_{i,n}) \\ H_{i,2}(y_{i,0}, y_{i,1}, ..., y_{i,n}) \\ . \\ . \\ . \\ H_{i,n}(y_{i,0}, y_{i,1}, ..., y_{i,n}) \end{pmatrix}. \qquad (23)$$

The formula (23) represents a system of linear algebraic equation, may be solved numerically. We will find the unique solution of this linear algebraic system (23) which corresponds to the unique solution of (21).

5 Numerical examples

In this section, we will introduce two examples as illustration for solving linear system of the Fredholm-Volterra integral equations.

Example 5.1.

Consider the linear Fredholm-Volterra integral equations

$$y(u,t+0.0002) = g(u,t) + \int_0^1 (uv)^2 y(v,t)dv + \int_0^t t\tau^2 y(u,\tau)d\tau. \qquad (24)$$

The exact solution is $y(u,t) = u^2+t$. If we divide the interval $[0,T],\ 0 \le T < 1$, as $0 = t_0 < t_1 < t_2 < t_3 = T$, where $t = t_i,\ i = 0,1,2,3$, and using the TR, the linear F-VIE (24) takes the form,

$$\begin{aligned}\delta_i y_{i,m} =& 0.0002u_m^2 + \sum_{j=0}^{i} \mu_j g_{j,m} + \sum_{j=0}^{i-1} \mu_j [\frac{1}{2}((t_j t_i)^2 - t_j^2) - 1] y_{j,m} \\ &+ \frac{\Delta v}{2} \sum_{j=0}^{i} \mu_j [(u_m v_0)^2 y_{j,0} + (u_m v_n)^2 y_{j,n} + 2\sum_{r=1}^{n-1} (u_m v_r)^2 y_{j,r}],\end{aligned}$$

where, $v_0 = 0,\ v_n = 1,\ v_r = \frac{r}{n},\ u_m = v_r,\ r = 1,2,...,9,\ j = 0,1,2,3$.
In table 5.1, we presented the absolute error $|y(u,t_i) - y_i(u)|, i = 0,1,2,3$, using the introduced numerical method TR with $N = 3$ in the interval $[0,0.6]$.

Table 5.1: The absolute error of solution of Eq.(24) by using TR with $N = 3$ and $0 \le T \le 0.6$.

u_m	$\lvert y(u_m,t_0) - y_0(u_m)\rvert$	$\lvert y(u_m,t_1) - y_1(u_m)\rvert$	$\lvert y(u_m,t_2) - y_2(u_m)\rvert$	$\lvert y(u_m,t_3) - y_3(u_m)\rvert$
0.1	1.99493×10^{-10}	1.12410×10^{-9}	6.03406×10^{-8}	7.09999×10^{-7}
0.2	7.97973×10^{-10}	1.12410×10^{-9}	6.03406×10^{-8}	7.09999×10^{-7}
0.3	1.80411×10^{-10}	1.12133×10^{-9}	6.03406×10^{-8}	7.09988×10^{-7}
0.4	3.19189×10^{-9}	1.12688×10^{-9}	6.03406×10^{-8}	7.09999×10^{-7}
0.5	4.99600×10^{-9}	1.12688×10^{-8}	6.03406×10^{-8}	7.09999×10^{-7}
0.6	7.21645×10^{-9}	1.12133×10^{-8}	6.03406×10^{-8}	7.09999×10^{-7}
0.7	9.76996×10^{-9}	1.13243×10^{-8}	6.03406×10^{-7}	7.09988×10^{-7}
0.8	1.27676×10^{-9}	1.12133×10^{-7}	6.03406×10^{-7}	7.10010×10^{-7}
0.9	1.62093×10^{-8}	1.11022×10^{-7}	6.03406×10^{-7}	7.10010×10^{-7}

Also, in figures 1.1-1.4, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method TR with different values of $t_i,\ i = 0,1,2,3$ with $N = 3$ in the interval $[0,1]$.

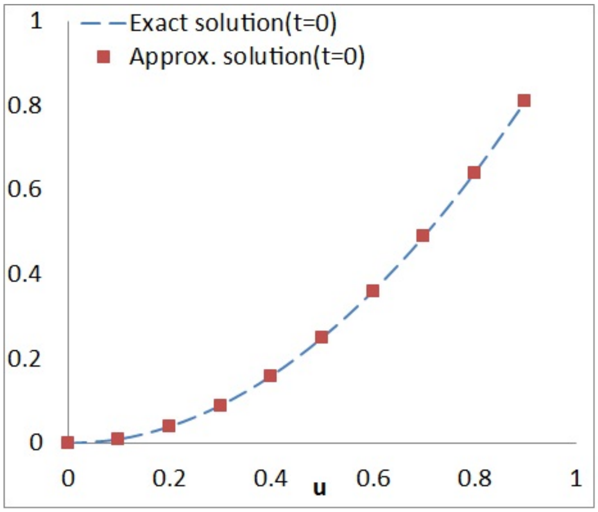

Figure 1.1.
Shows the exact solution $y(u, t_0) = u^2 + t_0$ and the approximate solution $y_0(u)$.

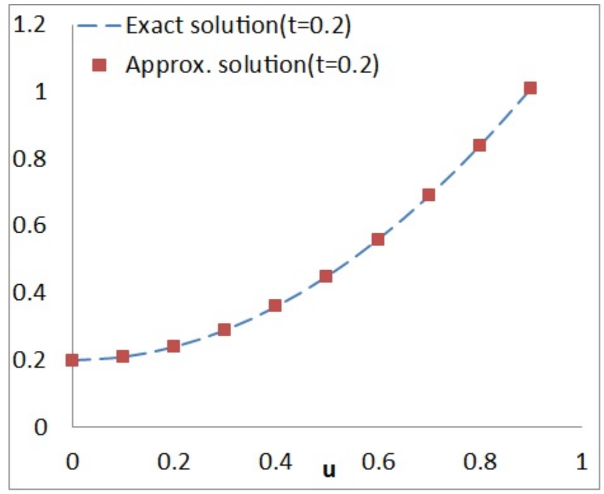

Figure 1.2.
Shows the exact solution $y(u, t_1) = u^2 + t_1$ and the approximate solution $y_1(u)$.

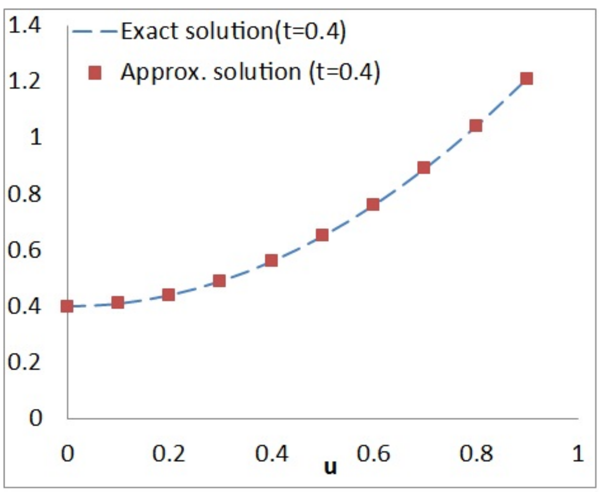

Figure 1.3.
Shows the exact solution $y(u, t_2) = u^2 + t_2$ and the approximate solution $y_2(u)$.

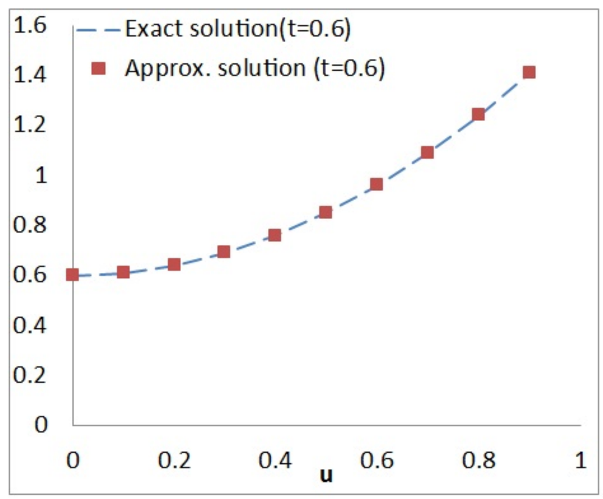

Figure 1.4.
Shows the exact solution $y(u, t_3) = u^2 + t_3$ and the approximate solution $y_3(u)$.

Example 5.2.

Consider the linear Fredholm-Volterra integral equations

$$y(u, t + 0.002) = g(u, t) + \int_0^1 e^{(u+v)} y(v, t) dv + \int_0^t \tau t^2 y(u, \tau) d\tau. \qquad (25)$$

The exact solution is $y(u, t) = e^u + t$. Using TR, the linear F-VIE (25) takes the form,

$$\delta_i y_i(u_m) = 0.002\ e^{u_m} + \sum_{j=0}^{i} \mu_j g_j(u_m) + \sum_{j=0}^{i-1} \mu_j [\frac{t_j}{3}((t_i)^3 - t_j^3) - 1] y_j(u_m)$$
$$+ \frac{\Delta v}{2} \sum_{j=0}^{i} \mu_j [e^{(u_m+v_0)} y_{j,0} + e^{(u_m+v_n)} y_{j,n} + 2 \sum_{r=1}^{n-1} e^{(u_m+v_r)} y_{j,r}].$$

In table 5.2, we presented the absolute error $|y(u,t_i) - y_i(u)|, i = 0,1,2,3,$ using the introduced numerical method TR with $N = 3$ in the interval $[0, 0.6]$.

Table 5.2: The absolute error of solution of Eq.(25) by using TR with $N = 3$ and $0 \le T \le 0.6$.

u_m	$\lvert y(u_m,t_0) - y_0(u_m)\rvert$	$\lvert y(u_m,t_1) - y_1(u_m)\rvert$	$\lvert y(u_m,t_2) - y_2(u_m)\rvert$	$\lvert y(u_m,t_3) - y_3(u_m)\rvert$
0.0	2.00000×10^{-9}	1.10022×10^{-8}	3.38518×10^{-7}	9.98201×10^{-7}
0.1	2.21034×10^{-9}	1.11022×10^{-8}	3.38618×10^{-7}	9.99201×10^{-7}
0.2	2.44281×10^{-9}	1.11022×10^{-8}	3.33067×10^{-7}	1.01030×10^{-7}
0.3	2.69972×10^{-9}	1.11022×10^{-8}	3.44169×10^{-7}	1.01030×10^{-7}
0.4	2.98365×10^{-9}	1.11022×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
0.5	3.29744×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
0.6	3.64424×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
0.7	4.02751×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
0.8	4.45108×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
0.9	4.91921×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}
10	5.43656×10^{-9}	1.11124×10^{-8}	3.33874×10^{-7}	1.01000×10^{-7}

Also, in figures 2.1-2.4, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method TR with different values of t_i, $i = 0,1,2,3$ with $N = 3$ in the interval $[0,1]$.

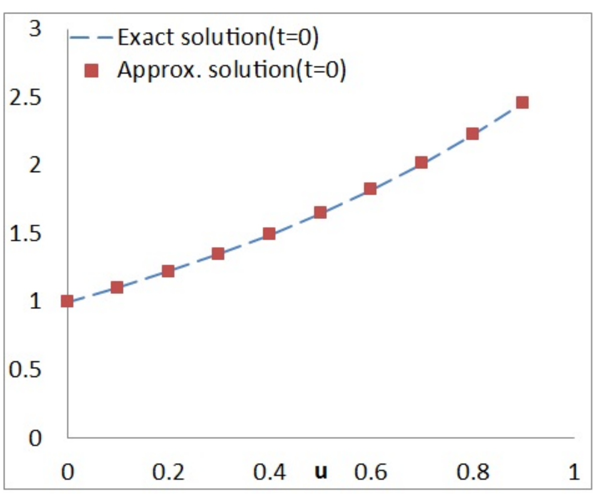

Figure 2.1.
Shows the exact solution $y(u,t_0) = e^u + t_0$ and the approximate solution $y_0(u)$.

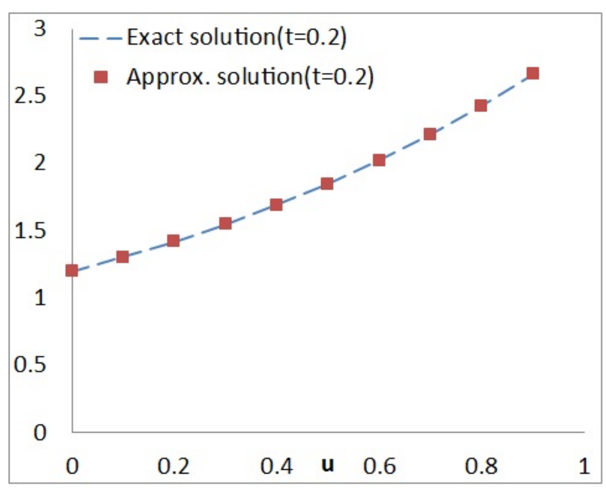

Figure 2.2.
Shows the exact solution $y(u,t_1) = e^u + t_1$ and the approximate solution $y_1(u)$.

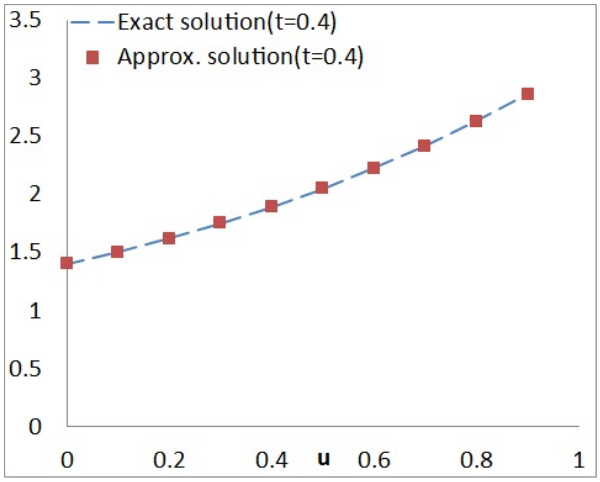

Figure 2.3.
Shows the exact solution $y(u, t_2) = e^u + t_2$ and the approximate solution $y_2(u)$.

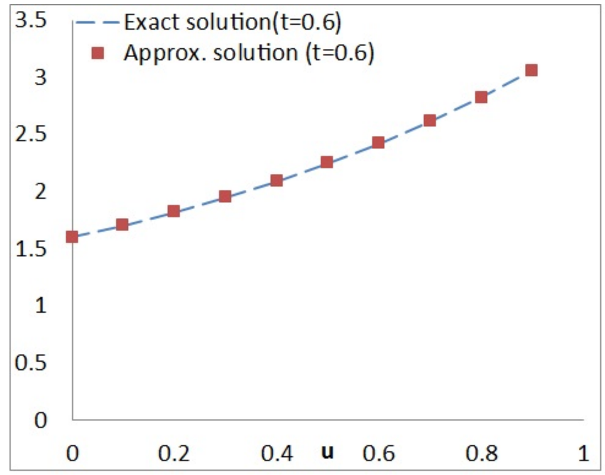

Figure 2.4.
Shows the exact solution $y(u, t_3) = e^u + t_3$ and the approximate solution $y_3(u)$.

6. Conclusions

From the above results and discussion, the following may be concluded:

1. The equation (5) has an unique solution $y(u, t)$ in the space $L_2[0, 1] \times C^n[0, T]$, under some conditions.

2. The mixed integral equation of the second kind, in time and position, after using quadratic method leads to a system of linear Fredholm integral equations of the second kind in position.

3. The system of linear Fredholm integral equations, using trapezoidal rule leads to a linear algebraic system.

4. If $q << 1$, we find that the numerical solution converges to the exact solution.

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