PYRAMIDAL Horn
Pyramidal Horn Antenna

Side View

Top View
Pyramidal Horn

Aperture Fields:

\[ E'_y(x', y') = E_0 \cos \left( \frac{\pi}{a_1} x' \right) e^{-j \left[ k \left( \frac{x'^2}{2 \rho_2} + \frac{y'^2}{2 \rho_1} \right) \right]} \]
(13-43a)

\[ H'_x(x', y') = -\frac{E_0}{\eta} \cos \left( \frac{\pi}{a_1} x' \right) e^{-j \left[ k \left( \frac{x'^2}{2 \rho_2} + \frac{y'^2}{2 \rho_1} \right) \right]} \]
(13-43b)
Radiated Fields:

\[
E_\theta = j \frac{kE_0 e^{-jkr}}{4\pi r} \left[ \sin \phi (1 + \cos \theta) I_1 I_2 \right] \tag{13-48b}
\]

\[
E_\phi = j \frac{kE_0 e^{-jkr}}{4\pi r} \left[ \cos \phi (1 + \cos \theta) I_1 I_2 \right] \tag{13-48c}
\]

\[
I_1 = \frac{1}{2} \sqrt{\frac{\pi \rho_2}{k}} \left\{ e^{j\frac{k_x^2 \rho_2}{2k}} \left[ \left( C(t'_2) - C(t'_1) \right) - j \left( S(t'_2) - S(t'_1) \right) \right] \right\} 
+ e^{j\frac{k_x^2 \rho_2}{2k}} \left[ \left( C(t''_2) - C(t''_1) \right) - j \left( S(t''_2) - S(t''_1) \right) \right] \tag{13-46}
\]

\[
I_2 = \sqrt{\frac{\pi \rho_1}{k}} e^{j\frac{k_y^2 \rho_1}{2k}} \left\{ \left[ \left( C(t_2) - C(t_1) \right) - j \left( S(t_2) - S(t_1) \right) \right] \right\} \tag{13-47}
\]
\[ \rho_1 = \rho_2 = 6\lambda \]
\[ a_1 = 5.5\lambda \]
\[ b_1 = 2.75\lambda \]
\[ a = 0.5\lambda \]
\[ b = 0.25\lambda \]

Fig. 13.19
\[ \rho_1 = \rho_2 = 6\lambda \]
\[ a_1 = 12\lambda \]
\[ b_1 = 6\lambda \]
\[ a = 0.5\lambda \]
\[ b = 0.25\lambda \]

Fig. 13.21
Pyramidal Horn Antenna

\[ E'_y(x', y') = E_0 \cos \left( \frac{\pi}{a_1} x' \right) e^{-j \left[ k \left( \frac{x'^2}{2\rho_2} + \frac{y'^2}{2\rho_1} \right) \right]} \]

Condition for Physical Realization:

\[ p_e = (b_1 - b) \left[ \left( \frac{\rho_e}{b_1} \right)^2 - \frac{1}{4} \right]^{1/2} \]
\[ p_h = (a_1 - a) \left[ \left( \frac{\rho_h}{a_1} \right)^2 - \frac{1}{4} \right]^{1/2} \]

\[ p_e = p_h \]
Pyramidal Horn: Design Procedure

Directivity of Pyramidal Horn Antenna can be obtained using Directivity curves for E-and H-Planes Sectoral Horn antenna

\[
G_0 \approx \frac{1}{2} \left( \frac{4 \pi}{\lambda^2} a_1 b_1 \right)
\]

\[
a_1 \approx \sqrt{3 \lambda \rho_2} \approx \sqrt{3 \lambda \rho_h}
\]

\[
b_1 \approx \sqrt{2 \lambda \rho_1} \approx \sqrt{2 \lambda \rho_e}
\]

\[
\rho_2 = \rho_h
\]

\[
\rho_1 = \rho_e
\]

\[
p_e = (b_1 - b) \sqrt{\left( \frac{p_e}{b_1} \right)^2 - \frac{1}{4}}
\]

\[
p_h = (a_1 - a) \sqrt{\left( \frac{p_h}{a_1} \right)^2 - \frac{1}{4}}
\]
Examples

1. $\rho_1 = \rho_2 = 6\lambda$, $a_1 = 5.5\lambda$, $b_1 = 2.75\lambda$, $a = 0.5\lambda$, $b = 0.25\lambda$

$$\rho_e = \sqrt{\rho_1^2 + \left(\frac{b_1}{2}\right)^2} = \sqrt{(6)^2 + \left(\frac{2.75}{2}\right)^2} \lambda = 6.1555\lambda$$

$$\rho_h = \sqrt{\rho_2^2 + \left(\frac{a_1}{2}\right)^2} = \sqrt{(6)^2 + \left(\frac{5.5}{2}\right)^2} \lambda = 6.6\lambda$$

$$p_e = (2.75 - 0.25)\lambda = \left[\left(\frac{6.1555}{2.75}\right)^2 - \frac{1}{4}\right]^{1/2} = 5.4544\lambda$$

$$p_h = (5.5 - 0.5)\lambda = \left[\left(\frac{6.6}{5.5}\right)^2 - \frac{1}{4}\right]^{1/2} = 5.4544\lambda$$

$$p_e = p_h = 5.4544\lambda$$
2. $\rho_1 = \rho_2 = 6\lambda$, $a_1 = 12\lambda$, $b_1 = 6\lambda$, $a = 0.5\lambda$, $b = 0.25\lambda$

\[
\rho_e = \sqrt{\rho_1^2 + \left(\frac{b_1}{2}\right)^2} = \sqrt{(6)^2 + (6/2)^2} \lambda = 6.7082\lambda
\]

\[
\rho_h = \sqrt{\rho_2^2 + \left(\frac{a_1}{2}\right)^2} = \sqrt{(6)^2 + (12/2)^2} \lambda = 8.4853\lambda
\]

\[
p_e = (6 - 0.25)\lambda = \left[\left(\frac{6.7082}{6}\right)^2 - \frac{1}{4}\right]^{1/2} = 5.75\lambda
\]

\[
p_h = (12 - 0.5)\lambda = \left[\left(\frac{8.4853}{12}\right)^2 - \frac{1}{4}\right]^{1/2} = 5.75\lambda
\]

$p_e = p_h = 5.75\lambda$
Design of Pyramidal horn antenna
Design Procedure

• To design a pyramidal horn, one usually knows the desired gain $G_0$ and the dimensions $a, b$ of the rectangular feed waveguide.

• The objective of the design is to determine the remaining dimensions ($a_1, b_1, pe, ph, Pe, and Ph$) that will lead to an optimum gain.
The design equations are derived by firsts electing values of \( a_1 \) and \( b_1 \) that lead to optimum directivities for the E- and H-plane sectoral horns.

\[
\begin{align*}
a_1 & \simeq \sqrt{3\lambda \rho_2} \\
b_1 & \simeq \sqrt{2\lambda \rho_1}
\end{align*}
\]

Since the overall efficiency (including both the antenna and aperture efficiencies) of a horn antenna is about 50%, the gain of the antenna can be related to its physical area. Thus it can be written by

\[
G_0 = \frac{1}{2} \frac{4\pi}{\lambda^2} (a_1 b_1) = \frac{2\pi}{\lambda^2} \sqrt{3\lambda \rho_2} \sqrt{2\lambda \rho_1} \simeq \frac{2\pi}{\lambda^2} \sqrt{3\lambda \rho_h} \sqrt{2\lambda \rho_e}
\]
since for long horns $\rho_2 \approx \rho_h$ and $\rho_1 \approx \rho_e$.

For a pyramidal horn to be physically realizable, $P_e$ and $P_h$ must be equal. Using this equality, it can be shown that gain reduces to

$$
\left( \sqrt{2\chi} - \frac{b}{\lambda} \right)^2 (2\chi - 1) = \left( \frac{G_0}{2\pi} \sqrt{\frac{3}{2\pi}} \frac{1}{\sqrt{\chi}} - \frac{a}{\lambda} \right)^2 \left( \frac{G_0^2}{6\pi^3} \frac{1}{\chi} - 1 \right)
$$

where

$$
\frac{\rho_e}{\lambda} = \chi \quad (13-56a)
$$

$$
\frac{\rho_h}{\lambda} = \frac{G_0^2}{8\pi^3} \left( \frac{1}{\chi} \right) \quad (13-56b)
$$

This Equation is the horn-design equation.
1. As a first step of the design, find the value of $\chi$ which satisfies (13-56) for a desired gain $G_0$ (dimensionless). Use an iterative technique and begin with a trial value of

$$\chi_{\text{trial}} = \chi_1 = \frac{G_0}{2\pi \sqrt{2\pi}}$$

2. Once the correct $\chi$ has been found, determine $\rho_e$ and $\rho_h$ using (13-56a) and (13-56b) respectively.

3. Find the corresponding values of $a_1$ and $b_1$.

$$a_1 = \sqrt{3\lambda \rho_2} \simeq \sqrt{3\lambda \rho_h} = \frac{G_0}{2\pi} \sqrt{\frac{3}{2\pi \chi}} \lambda$$

$$b_1 = \sqrt{2\lambda \rho_1} \simeq \sqrt{2\lambda \rho_e} = \sqrt{2\chi \lambda}$$
Problems of horn antenna
Example 13.6:

**Given:** X-Band (8.2-12.4 GHz), $f = 11$ GHz Horn; Gain=22.6 dB

\[ a = 0.9 \text{ in (2.286 cm)}, \quad b = 0.4 \text{ in (1.016 cm)} \]

**Find:** Dimensions Of Pyramidal Horn

**Solution**

\[ G_0 (dB) = 22.6 = 10 \log_{10} G_0 \quad \Rightarrow \quad G_0 = 10^{2.26} = 181.97 \]

At $f = 11$ GHz \( \Rightarrow \lambda = \frac{30 \times 10^9}{11 \times 10^9} = 2.7273 \text{ cm} \)

\[ b = \frac{1.016}{2.7273} \lambda = 0.3725 \lambda; \quad a = \frac{2.286}{2.7273} \lambda = 0.8382 \lambda \]
1. Initial value of $\chi$

$$\chi_1 = \frac{G_0}{2\pi \sqrt{2\pi}} = \frac{181.97}{2\pi \sqrt{2\pi}} = 11.5539$$

which does not satisfy (12-56), or

$$\left( \sqrt{2\chi} - \frac{b}{\chi} \right)^2 (2\chi - 1) = \left( \frac{G_0}{2\pi} \sqrt{\frac{3}{2\pi}} \frac{1}{\sqrt{\chi}} - \frac{a}{\lambda} \right)^2 \left( \frac{G_0^2}{6\pi^3} \frac{1}{\chi} - 1 \right)$$

After few tries, a more accurate value is

$$\chi = 11.1157$$

2. $\rho_e = \chi\lambda = 11.1157\lambda = 30.316 \text{ cm} = 11.935 \text{ in.}$

$$\rho_h = \frac{G_0^2}{8\pi^3} \left( \frac{1}{\chi} \right) \lambda = 12.0094\lambda = 32.753 \text{ cm} = 12.895 \text{ in.}$$
3. \[ a_1 = \sqrt{3\lambda \rho_2} \approx \sqrt{3\lambda \rho_h} = \frac{G_0}{2\pi} \sqrt{\frac{3}{2\pi \chi}} \lambda = 6.002 \lambda \]

\[ = 16.370 \text{ cm} = 6.445 \text{ in.} \]

\[ b_1 = \sqrt{2\lambda \rho_1} \approx \sqrt{2\lambda \rho_e} = \sqrt{2 \chi \lambda} = 4.715 \lambda \]

\[ = 12.859 \text{ cm} = 5.063 \text{ in.} \]

4. \[ p_e = (b_1 - b) \left[ \left( \frac{p_e}{b_1} \right)^2 - \frac{1}{4} \right]^{1/2} = 10.005 \lambda \]

\[ = 27.286 \text{ cm} = 10.743 \text{ in.} \]

\[ p_h = (a_1 - a) \left[ \left( \frac{p_h}{a_1} \right)^2 - \frac{1}{4} \right]^{1/2} = 10.005 \lambda \]

\[ = 27.286 \text{ cm} = 10.743 \text{ in.} \]
13.21. Design a pyramidal horn antenna with optimum gain at a frequency of 10 GHz. The overall length of the antenna from the imaginary vertex of the horn to the center of the aperture is 10\( \lambda \) and is nearly the same in both planes. Determine the

(a) Aperture dimensions of the horn (in cm).
(b) Gain of the antenna (in dB)
(c) Aperture efficiency of the antenna (in %). Assume the reflection, conduction, and dielectric losses of the antenna are negligible.
(d) Power delivered to a matched load when the incident power density is 10 \( \mu \) watts/m\(^2\).
\( \lambda = \frac{30 \times 10^9}{10 \times 10^9} = 3 \text{ cm} \)

a. \( a_1 \approx \sqrt{3 \lambda^2} = \sqrt{3 \cdot 5.477 \lambda^2} = 5.477 \lambda = 5.477 \times 13.416 = 73.18 \text{ cm} \)

b. \( b_1 \approx \sqrt{2 \lambda^2} = \sqrt{2 \cdot 5.477 \lambda} = 4.472 \lambda = 4.472 \times 13.416 = 59.84 \text{ cm} \)

c. \( G_0 = \frac{1}{2} \frac{4\pi}{\lambda^2} (a_1 b_1) = \frac{1}{2} \frac{4\pi}{\lambda^2} (5.477 \lambda)(4.472 \lambda) = 53.89 = 21.87 \text{ dB} \)

d. \( \varepsilon_R \varepsilon_{cd} \varepsilon_{ap} = 1 \cdot 1 \cdot \varepsilon_{ap} = \frac{1}{2}, \quad \varepsilon_{ap} = \frac{1}{2} = 50\% \)

\( \text{d.} \quad A_{em} = \frac{\lambda^2}{4\pi} G_0 = \frac{3^2}{4\pi} (153.89) = 110.2156 \text{ cm}^2 = 110.2156 \times 10^{-4} \text{ m}^2 \)

\( P_{rec} = W A_{em} = 10 \times 10^{-6} \times 110.2156 \times 10^{-4} = 1.102156 \times 10^{-10} = 11.02156 \times 10^{-8} \)

\( P_{rec} = 11.02156 \times 10^{-8} = 0.1102156 \mu \text{ Watts} \)
$a_1 = \frac{G_0}{2\pi} \sqrt{\frac{3}{2\pi^2}} \lambda = \frac{50.7}{2\pi} \sqrt{\frac{3}{2\pi (2.96795)}} \lambda = 3.23646 \lambda = 8.8268 \text{ cm} = 3.475''$

$b_1 = \sqrt{2} \lambda = \sqrt{2 (2.96795)} \lambda = 2.43637 \lambda = 6.6447 \text{ cm} = 2.616''$

$P_e = (b_1 - b) \left[ (\frac{pe}{b_1})^2 - \frac{1}{4} \right]^{1/2} = 6.25263 \text{ cm} = 2.46167''$

$P_h = (a_1 - a) \left[ (\frac{ph}{a_1})^2 - \frac{1}{4} \right]^{1/2} = 6.25269 \text{ cm} = 2.46169'' \Rightarrow P_e \approx P_h \approx 6.2526 \text{ cm} \approx 2.4617''$
A standard-gain X-band (8.2–12.4 GHz) pyramidal horn has dimensions of $\rho_1 \approx 13.5$ in. (34.29 cm), $\rho_2 \approx 14.2$ in. (36.07 cm), $a_1 = 7.65$ in. (19.43 cm), $b_1 = 5.65$ in. (14.35 cm), $a = 0.9$ in. (2.286 cm), and $b = 0.4$ in. (1.016 cm).

(a) Check to see if such a horn can be constructed physically.

\[
\begin{align*}
\rho_1 &= 13.5'' = 34.49 \text{ cm} \\
\rho_2 &= 14.2'' = 36.07 \text{ cm} \\
a_1 &= 7.65'' = 19.43 \text{ cm} \\
b_1 &= 5.65'' = 14.35 \text{ cm} \\
a &= 0.9'' = 2.286 \text{ cm} \\
b &= 0.4'' = 1.016 \text{ cm}
\end{align*}
\]

\[
\begin{align*}
\rho_e &= \left[ (\rho_1^2 + (b_1/2)^2) \right]^{1/2} = 13.7924'' = 35.0327 \text{ cm} \\
\rho_h &= \left[ (\rho_1^2 + (a_1/2)^2) \right]^{1/2} = 14.7061'' = 37.3536 \text{ cm}
\end{align*}
\]

\[
\begin{align*}
\text{a. } P_e &= (b_1-b) \sqrt{\frac{(P_e)^2}{b_1}} = 5.65 - 0.4 \sqrt{\left(\frac{13.7924}{5.65}\right)^2 - \frac{1}{4}} = 12.544'' = 31.862 \text{ cm} \\
\text{b. } P_h &= (a_1-a) \sqrt{\frac{(P_h)^2}{a_1}} = 7.65 - 0.9 \sqrt{\left(\frac{14.7061}{7.65}\right)^2 - \frac{1}{4}} = 12.529'' = 31.8246 \text{ cm}
\end{align*}
\]

Therefore $P_e = P_h$, and the pyramidal horn is physically realizable.