The Logic of Tied Implications, Part 1: Properties, Applications and Representation

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Abstract

A conjunction T *ties* an implication operator A if the identity $A(a, A(b, z)) = A(T(a, b), z)$ holds (Fuzzy Sets and Systems 136 (2003) 291-311). We study the class of *tied adjointness algebras* (which are five-connective algebras on two partially ordered sets), in which the implications are tied by triangular norms. This class contains, besides residuated implications, several other implications employed in fuzzy logic. Nevertheless, we show that the algebraic inequalities of residuated algebras remain true for our tied implications, but in forms that distribute roles over the five connectives of the algebra.

We apply the properties of tied implications to a generalized modus ponens inference scheme with two successive rules. We prove its equivalence to a scheme with one compound rule, when both schemata are interpreted by the compositional rule of inference, and all connectives are taken from one tied adjointness algebra. Then we quote another application of this rich theory, a notion of many-valued rough sets, which exhibit the basic mathematical behaviour of the rough sets of Pawlak.

A comparator H is said to be *prelinear* if it satisfies $H(y,z) \vee H(z,y) = 1$ for all y, z (Hájek). We introduce prelinear tied adjointness algebras, in which two comparators are prelinear. We provide a representation of those algebras, as subdirect products of tied adjointness chains, on the lines of Hájek's representation of BL-algebras. But our representations are more economical, because we employ minimal prime filters (on residuated lattices) only; rather than all prime filters.

Keywords: Nonclassical logics; Connectives; Tied implication; Generalized modus ponens; Many-valued rough sets; Prelinearity; Subdirect product

1 Introduction

Fuzzy logic is the logic of vagueness. It provides a mathematical machinery to handle partial degrees of truth of vague propositions. We submit to the reader a new framework for the algebraic study of logical connectives; the class of what we call *tied adjointness algebras*. This framework couples a strong set of theorems with a wide variety of logical connectives frequently met in the literature.

Our work is founded in the well-established domain of residuated lattices, whereby implications and conjunctions are related by residuation (= adjointness). We build on the work [1] of Abdel-Hamid and Morsi, in which they have enriched adjointness structures with one more conjunction, this time a binary operation T (called a *tying-conjunction*) that *ties* an implication A in the following sense:

$$\forall a, b, z : \qquad A(T(a,b), z) = A(a, A(b,z)).$$

This property extends to multiple-valued logic the following equivalence in classical logic: $((X \& Y) \Longrightarrow Z) \equiv (X \Longrightarrow (Y \Longrightarrow Z))$. It holds for several types of implications used in fuzzy logic.

In the practice of fuzzy logic, one sometimes needs to combine different types of conjunctions and implications. Höhle and Šostak [54] base fuzzy topologies on complete quasi-monoidal lattices over quantales, which combine several conjunctions and implications. Also, Hájek [45, Section 9.1] elaborated a logic having three conjunctions and three implications; motivated by a similar structure of Takeuti and Titani [81]. Our work falls in this line of research, but with the feature of the tiedness relation between T and A.

Section 2 offers needed background on the notion of adjointness. From the very start, we follow Morsi [66] in proposing the simultaneous use of two partially ordered sets (posets) in one adjointness algebra. We present in Subsection 3.2 our arguments for this approach, in the light of the examples of Subsection 3.1. However, in the special case of a residuated algebra (Subsection 2.2), only one poset can be used, because the conjunction there has a two-sided identity element.

In Section 3, we supply our formal definition of tied adjointness algebras. We request tying-conjunctions to be commutative integral ordered monoid operations, which we simply call *triangular norms*, and we require them to have residuated implications. We put forward arguments for these stipulations, both logical and practical. Then we try to validate our claim that our two-posets structures admit an abundance of useful inequalities, yet encompass a diverse assortment of examples.

We illustrate the potential usefulness of tied adjointness algebras through two applications. In the first one (Section 4), we use the connectives of a tied adjointness algebra to interpret Generalized Modus Ponens (**GMP**) inference schemata, in the vein of the Compositional Rule of Inference of Zadeh. We show that those interpretations of **GMP** satisfy the intuitive criterion that every inference scheme with a compound rule is equivalent to another inference scheme with a succession of simple rules.

Another application is quoted from [66] in Section 5. This is a notion of (L, P)-valued rough sets, whereby a P-valued partition of a set V induces a coarse classification of L^V. Those enjoy extensions of some basic behaviour of the rough sets of Pawlak [75].

In Section 6 we define *prelinear tied adjointness algebras* (over pairs of lattices), in analogy with the prelinear structures of Esteva, Godo, Hájek and Höhle. But in our algebras, two comparators have to be prelinear, in order to possess an adequate representation. Our results are analogous to the ones obtained for **BL** in [45] and **MTL** in [26]. They include a representation of those algebras as lattice-subdirect products of tied adjointness chains. Our representations are economical ones, because we employ minimal prime filters only; rather than all prime filters.

Our conclusions from this part wind it up in Section 7.

Throughout the present part, we use combinatorial notations for logical connectives, in order to distinguish them clearly from the corresponding connectives in syntax, for which we shall use infix notations. We remark that both types of notation of logical connectives are found in the literature on fuzzy logic and its applications. By using the term *algebra* for our two-posets structures, we slightly abuse this term, particularly also that we do not always request partial orders to come from lattice operations. However, we prefer this term, because our discussions are basically algebraic. This abuse is not unprecedented; remember *linear associative algebras*, with their vectors and scalars, and possibly their orders.

2 Background

For the sake of simplification, we begin in Section 2, 3 by discoursing in the more general setting of adjointness algebras on partially ordered sets, possibly without the lattice structure. We shall reintroduce the lattice operations starting from Section 4.

2.1 Adjointness algebras

Throughout, $(P, \leq_P)$ and $(L, \leq_L)$ denote partially ordered sets (*posets*). P has a top element, denoted by 1. But L need not have a top element. Lowercase Latin letters from the end of the alphabet (e.g. "x", "y" and "z") will be used as variables ranging over the elements of L, and lowercase Latin letters from the beginning of the alphabet (e.g. "a", "b" and "c") will be used as variables ranging over the elements of P.

Definition 2.1 *([66], cf. [64]) An* adjointness algebra *is an 8-tuple* $(L, \leq_L, P, \leq_P, 1, A, K, H)$*, in which* $(L, \leq_L)$*,* $(P, \leq_P)$ *are two posets with a top element* 1 *for* $(P, \leq_P)$*, and the following four conditions are satisfied:*

(i) The operation $A : P \times L \to L$ *is antitone in the left argument and monotone in the right argument, and it has* $1 \in P$ *as a left identity element. We call* A *an* implication *on* (L, P)*.*

(ii) The operation $K : P \times L \to L$ *is monotone in each argument and has* $1 \in P$ *as a left identity element. We call* K *a* conjunction *on* (L, P)*.*

(iii) The operation $H : L \times L \to P$ *is antitone in the left argument and monotone in the right argument, and it satisfies*

$$\forall y, z \in L : \qquad H(y, z) = 1 \quad \text{iff} \quad y \leq_L z. \tag{1}$$

We call H *a* comparator *on* L *(it is called a* forcing-implication *in [1], [64]).*

(iv) The three operations A*,* K *and* H *are mutually related by the following condition,* $\forall a \in P, \ \forall y, z \in L :$

$$\textbf{\textit{Adjointness:}} \quad y \leq_L A(a, z) \quad \text{iff} \quad K(a, y) \leq_L z \quad \text{iff} \quad a \leq_P H(y, z)\,.$$

We call the ordered triple (A, K, H) *an* implication triple *on* (L, P)*. (In the case* $P = L$*, we say "on* P*" to mean "on* (P, P)*".)*

An adjointness lattice *is an adjointness algebra whose two underlying posets are lattices. A* complete adjointness lattice *is an adjointness algebra over two complete lattices. An* adjointness chain *is an adjointness lattice whose two orders are linear.*

Note that Definition 2.1, as well as Definition 3.1 below, differ from the original ones of [1] and [64] in that those assume one poset only (that is, $P = L\}$. See Subsection 3.2 below. Nevertheless, most results in [1] and [64] remain valid for our more general definitions, with virtually duplicate proofs.

The subject of a possibly noncommutative, nonassociative conjunction K with two, possibly distinct, implication-like adjoints (here, A and H, which are distinct whenever K is noncommutative) is an old one. Early works include Dilworth [14], [15] and Pavelka [74]. It is currently the subject of extensive research, establishing it as an essential trend in multiple-valued logics and fuzzy logic. There, results from algebraic logic, residuation theory and the theory of topological semigroups (cf. [73], [79]) were connected in order to obtain new models of multiple-valued logic, and new connectives on fuzzy statements. This research materialized in numerous articles and books, among which [1], [2], [3], [4], [6], [7], [8], [9], [12], [13], [17], [18], [19], [20], [21], [22], [24], [25], [26], [27], [29], [31], [32], [34], [41], [42], [43], [44], [45], [46], [47], [49], [50], [51], [52], [53], [55], [56], [58], [59], [60], [61], [62] [64], [65], [66], [67], [68], [72], [76], [82], [85], [86], [87], [88], [89], and others, are concerned with residuation in the multiple-valued-logic component of fuzzy logic. **Adjointness** lies also at the basis of algebraic logic and substructural logics [16], [77], [78]. In [16], one finds a detailed algebraic study of adjointness structures under the name *residuated partially ordered groupoids*. Dunn has carried out a general analysis of adjointness structures, under the title of *Partial Gaggles*, see [16, Chapter 4]. On the one hand, the condition **Adjointness** is a main tool in building a useful calculus for implications; generating universally valid inequalities. On the other hand, it is now known that the membership values have different semantics, which frequently coexist in the same application [23]. It is natural to admit noncommutative, nonassociative conjunctions between truth values of differing semantics. Similar argument restricts the need for the comparator axiom (1).

Lemma 2.1 *([66], cf. [1]). Let $(L, \leq_L, P, \leq_P, 1, A, K, H)$ be an adjointness algebra. Then*

(i) For all a in P and z in L:

$$\begin{aligned} A(a,z) &= \sup\{y \in L \mid \; K(a,y) \leq_L z\} \\ &= \sup\{y \in L \mid \; a \leq_P H(y,z)\}. \end{aligned}$$

(This means that these particular suprema always exist in $(L, \leq_L)$.). Similarly, K is derived from A, H, and H is derived from A, K.

(ii)

$$A\left(\sup_j a_j, \inf_m z_m\right) = \inf_{j,m} A\left(a_j, z_m\right), \tag{2}$$

$$K\left(\sup_j a_j, \sup_m y_m\right) = \sup_{j,m} K\left(a_j, y_m\right), \tag{3}$$

$$H\left(\sup_j y_j, \inf_m z_m\right) = \inf_{j,m} H\left(y_j, z_m\right), \tag{4}$$

for all indexed families $\{a_j\}$ *in* P *and* $\{z_m\}, \{y_m\}$ *in* L*, such that the suprema and infima in the left-hand sides exist. Also,*

$$H\left(\sup_j y_j, \sup_j z_j\right) \geq_P \inf_j H\left(y_j, z_j\right), \tag{5}$$

$$H\left(\inf_j y_j, \inf_j z_j\right) \geq_P \inf_j H\left(y_j, \gamma_j\right). \tag{6}$$

(iii) In the case when $L = P$*,* $A = H$ *if and only if* K *is commutative.*

(iv) If P *has a bottom element* 0*, then* L *is bounded, and* $A(0, z) = 1$*,* $K(0, y) = 0, \quad \forall z, y \in L$*.*

Whenever P and L are complete, a given implication A becomes the first component in a (unique) implication triple if and only if it satisfies the condition (2). A similar situation applies to K and H [1]. Accordingly, we consider only implications that satisfy (2) and conjunctions that satisfy (3).

Examples of adjointness algebras are deferred to Section 3, in order to minimize unnecessary repetitions.

2.2 Residuated algebras

We can speak of right identity, associativity or commutativity of a conjunction K only when $(P, \leq_P)$ equals $(L, \leq_L)$. Recall that $K : L \times L \to L$ is said to be *associative* if it satisfies:

$$\forall x, y, z \in L : \quad K(x, K(y, z)) = K(K(x, y), z),$$

and *commutative* when it satisfies:

$$\forall x, y, \in L : \quad K(x, y) = K(y, x).$$

We can rephrase the well-known definition of residuated algebras to read

Definition 2.2 *A* residuated algebra *is a special case of adjointness algebras, in which the conjunction is both commutative and associative.*

The two adjoints of a commutative conjunction coincide. Therefore, a residuated algebra always takes the form $(P, \leq_P, P, \leq_P, I, T, I)$, in which T is a commutative and associative conjunction, and (I, T, I) is an implication triple on P. We shall usually simplify this notation to $(P, \leq_P, T, I)$.

The properties required from such T make the triple $(P, \leq_P, T)$ a *commutative integral ordered monoid*. However, we prefer in this work to simplify this terminology and call T a *triangular norm* on $(P, \leq_P)$, and we refer to its two-sided adjoint I as its *residuated implication* (its *R-implication*). This usurps the terminology of Schweizer and Sklar [79], whereby the term triangular norm is reserved for commutative and associative conjunctions on the unit interval $[0, 1]$ of real numbers, only. However, we still mark this important special case by using the term *t-norm* for triangular norms on $[0, 1]$ only.

Note that our triangular norms are required to possess residuated implications, see Subsection 2.1. Thus, when $P = [0, 1]$, our t-norms have to be left-continuous, in order to satisfy (3). See [55], [58] for general methods for their construction.

We shall use the term **Residuation** for the form taken by the condition **Adjointness** in the case of a residuated algebra.

Example 2.1 *The following are three basic continuous t-norms on the unit interval, together with their residuation implications:*

(i) Łukasiewicz t-norm*:* $T(x, y) = \max\{0, x + y - 1\}$,
Łukasiewicz implication*:* $I(x, z) = \min\{1, z + 1 - x\}$.

(ii) Gödel t-norm*:* $T(x, y) = \min\{x, y\}$,
Gödel implication*:* $I(x, z) = \begin{cases} 1, & x \leq z \\ z, & x > z \end{cases}$.

(These two connectives are generalized over arbitrary complete Heyting algebras.)

(iii) Product (Goguen) t-norm*:* $T(x, y) = xy$ *(product of reals) [40],*
Goguen implication*:* $I(x, z) = \begin{cases} 1, & x \leq z \\ \frac{z}{x}, & x > z \end{cases}$.

The nilpotent minimum *[31] is a left-continuous, but not continuous, t-norm. It and its residuum are defined by:*

$$T(x, y) = \begin{cases} 0 & x + y \leq 1 \\ \min(x, y) & x + y > 1 \end{cases},$$

$$I(x, z) = \begin{cases} 1 & x \leq z \\ \max\{1 - x, z\} & x > z \end{cases}.$$

The basic inequalities in residuated algebras are hinted about at the end of Section 3. Then we list them again in syntax in Part 2 [69].

3 Tied adjointness algebras

3.1 Definition and examples

Abdel-Hamid and Morsi have enriched adjointness algebras with one more conjunction, as we next quote:

Definition 3.1 *(Morsi [66], cf. Abdel-Hamid and Morsi [1]) A binary operation D on P is said to* tie *an operation $F : P \times L \to L$ if the following identity holds $\forall a, b \in P, \forall z \in L$:*

$$F(a, F(b, z)) = F(D(a, b), z), \tag{7}$$

and we say that F is tied *by D.*

The property of *being tied* can be seen as a weakened form of associativity, particularly when the operation F is a conjunction.

Definition 3.2 *(Cf. [82]) $F : P \times L \to L$ is said to satisfy the* exchange principle *if*

$$\forall a, b \in P, z \in L : \quad F(a, F(b, z)) = F(b, F(a, z)). \tag{8}$$

Definition 3.3 *([66], cf. [1]) $F : P \times L \to L$ is said to* distinguish left arguments *if for each distinct pair a and b in P there is $z \in L$ such that $F(a, z) \neq F(b, z)$.*

The *symmetric* of a binary operation D on P is the binary operation D' defined by $\forall a, b \in P : \qquad D'(a, b) = D(b, a)$.

Theorem 3.1 *([66], cf. [1]) In an adjointness algebra $(L, \leq_L, P, \leq_P, 1, A, K, H)$:*

(i) A binary operation on P ties A if and only if its symmetric ties K.

(ii) The implication A satisfies the exchange principle if and only if the conjunction K satisfies it [64].

Suppose further that $(P, \leq_P)$ is a complete lattice.

(iii) The implication A distinguishes left arguments if and only if the conjunction K does so.

(iv) *If a binary operation on P ties A, then there is a greatest binary operation T_A on P that ties A. This T_A is monotone and associative. (This is the reason why tied implications are also said to be* associatively tied *[1].)*

(v) *The operation T_A becomes commutative if and only if A satisfies the exchange principle, if and only if K satisfies the exchange principle.*

(vi) *If A is tied and distinguishes left arguments, then T_A becomes the unique binary operation tying A. In this case, T_A is supremum-preserving (that is, it satisfies identity (3)), and 1 is a two-sided identity element for it.*

(vii) *Consequently, if A is tied, satisfies the exchange principle and distinguishes left arguments, then it is tied by a supremum-preserving triangular norm on P.*

Definition 3.4 *([66], cf. [1]) A* tied adjointness algebra *is an algebra*

$$\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$$

in which $(L, \leq_L, P, \leq_P, 1, A, K, H)$ is an adjointness algebra, $(P, \leq_P, T, I)$ is a residuated algebra, and T ties A; that is,

$$\forall a, b \in P, \ \forall z \in L : \qquad A(T(a, b), z) = A(a, A(b, z)). \tag{9}$$

The class of all tied adjointness algebras is denoted by ADJT.

The preceding definition differs from the more general Definition 3.1 in that it requests the tying operation T to be a triangular norm. Our reasons for this extra specification are explained in Remark 3.1 below, after we list various types of tied adjointness algebras in the next example.

During the process of submitting this paper for publication, we were told by an anonymous referee that an initial chapter in the Ph.D. Thesis [37] of Sujata Ghosh (2004) has introduced and discussed logics of posets with monoidal operator and residuation, which were also reported in a seminar presentation [38] in the year 2001. We quote the learned referee on this, since we have not seen those two works. His information does not make it clear whether they include a notion similar to (9) or not. But we add that the first systematic study of tying conjunctions has been carried out by Abdel-Hamid and Morsi in [1], an extended summary of which has been included in the Proceedings of the Seventh International Conference on Fuzzy Theory and Technology, Atlantic City, in the year 2000 (pages 49-51).

Example 3.1 *(i)* *Since a triangular norm is associative, it ties itself, and so it ties its residuated implication I. Therefore, all residuated algebras $(P, \leq_P, T, I)$ are tied adjointness algebras $(P, \leq_P, I, T, I, T, I)$. In fact,* *triangular norms exhaust all commutative tied conjunctions, whereas the R-implications exhaust all tied implications that are also comparators and satisfy the exchange principle.*

(ii) Let $\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$ *be a tied adjointness algebra, and suppose a strong negation (an order-reversing involution) n exists on* $(L, \leq_L)$*. Then the* S-image *of this* Λ *under* n *is the tied adjointness algebra* $\Lambda_S = (L, \leq_L, P, \leq_P, A_S, K_S, H_S, T, I)$ *defined by:*

$$\begin{aligned} A_S(a,z) &= n(K(a, n(z))), \qquad a \in P, z \in L, \\ K_S(a,y) &= n(A(a, n(y))), \qquad a \in P, y \in L, \end{aligned}$$

which is not associative for many choices of Λ*, and*

$$H_S(y,z) = H(n(z), n(y)), \qquad y, z \in L;$$

that is, H_S *is the* n*-contrapositive comparator of* H*.*

This S-bijection on ADJT is self-inverse. A_S *becomes the usual S-type implication of a triangular norm* T *when* Λ *is the residuated algebra of* T*.*

(In (i) and (ii), a generalization can readily be achieved by dropping the requirement of commutativity of T*.)*

(iii) The Yager implication *[88] on* $[0,1]$ *is given by*
$A_Y(a,c) = \begin{cases} 1, & a = c = 0 \\ c^a, & otherwise \end{cases}.$

It is tied only by the Goguen t-norm Product. Its implication triple is completed as follows:
$K_Y(a,b) = \begin{cases} 0, & ab = 0 \\ b^{\frac{1}{a}}, & otherwise \end{cases},$
which is not associative, and
$H_Y(b,c) = \begin{cases} 1, & b \leq c \\ \frac{\ln b}{\ln c}, & b > c > 0 \\ 0, & b > c = 0 \end{cases}.$

(iv) ([66], cf. [64]) If L and P are bounded, then the smallest implication on (L,P) *is*

$$A_\circ(a,z) = \begin{cases} 1, & a = 0 \\ z, & otherwise \end{cases}. \qquad (10)$$

This $A_\circ$ *is called the* stringent implication *on* (L,P)*. The greatest conjunction on* (L,P)*, called the* extravagant conjunction*, is*

$$K_\circ(a,y) = \begin{cases} 0, & a = 0 \\ y, & otherwise \end{cases}. \qquad (11)$$

The smallest P-valued comparator on L is the Gaines-Rescher implication *[36]:*

$$H_{\circ}(y,z)=\begin{cases}1, & y\leq_L z\\ 0, & y\nleq_L z\end{cases}. \tag{12}$$

The tuple $(L,\leq_L,P,\leq_P,A_{\circ},K_{\circ},H_{\circ})$ is an adjointness algebra. $A_{\circ}$ and $K_{\circ}$ are tied by any binary operation D on P that satisfies $D(a,b)\neq 0$ for all nonzero a,b in P. Note that $A_{\circ}$ satisfies the exchange principle, but it does not distinguish left arguments unless P is the doubleton lattice $\{0,1\}$. Nevertheless, whenever P is a chain, the Gödel triangular norm Min ties $A_{\circ}$. Also, in the case $L=P$, $K_{\circ}$ is associative, and so it ties itself.

(v) The associative, noncommutative conjunction $K:[0,1]^2\longmapsto[0,1]$ defined as follows:

$$K(a,b)=\begin{cases}0 & 2a+b\leq 1\\ \min\{a,b\} & \textit{otherwise}\end{cases}$$

distinguishes left arguments, but does not satisfy the exchange principle. It is direct to see that the implication triple for this conjunction is given by:

$$A(a,c)=\begin{cases}1 & a\leq c\\ \max\{1-2a,c\} & a>c\end{cases},$$

and

$$H(b,c)=\begin{cases}1 & b\leq c\\ \max\{(1-b)/2,c\} & b>c\end{cases}.$$

As K ties itself, its symmetric K' is the unique binary operation that ties A.

(vi) Let $(L,\leq)$ be a complete lattice, and let $\mathfrak{s}(L)$ be the lattice of all supremum-preserving lower functions on L (i.e., functions less or equal to id_L). This $\mathfrak{s}(L)$ is a complete lattice under the usual partial order of functions, with top element id_L and bottom element the constant function 0. Define $K:\mathfrak{s}(L)\times L\to L$ by $K(g,\mu)=g(\mu)$, $(g,\mu)\in\mathfrak{s}(L)\times L$. This K is a supremum-preserving conjunction. It distinguishes left arguments, but does not satisfy the exchange principle. Its implication triple is given by:

$A(g,\xi)=\sup\{\mu\in L \mid g(\mu)\leq\xi\}, \quad (g,\xi)\in\mathfrak{s}(L)\times L,$

$H(\mu,\xi)=\sup\{g\in\mathfrak{s}(L) \mid g(\mu)\leq\xi\}, \quad \mu,\xi\in L,$

and so $H(\mu,\xi)(\eta)=\begin{cases}\eta\wedge\xi, & \eta\leq\mu\\ \eta, & \eta\nleq\mu\end{cases}, \quad \mu,\xi,\eta\in L.$

Composition of functions $\circ$ is a non-commutative triangular norm on $\mathfrak{s}(L)$, and it ties K.

Example 3.2 *Modifying slightly the Yager implication, we obtain the following implication triple* $(\breve{A}, \breve{K}, \breve{H})$ *on* $[0,1]$:

$$\breve{A}(a,c) = \begin{cases} 1, & a=0 \\ c^{\frac{1+a}{2}}, & a>0 \end{cases},$$

$$\breve{K}(a,b) = \begin{cases} 0, & a=0 \\ b^{\frac{2}{1+a}}, & a>0 \end{cases},$$

$$\breve{H}(b,c) = \begin{cases} 1, & b \le c \\ 2^{\frac{\ln b}{\ln c}} - 1, & c < b \le \sqrt{c} \\ 0, & \sqrt{c} < b \end{cases}.$$

By direct verification, it is seen that this adjointness algebra cannot be tied by any binary operation on $[0,1]$.

Remark 3.1 *Definition 3.4 differs from Definition 3.1, in that it requires tying conjunctions to satisfy the five properties: monotonicity, associativity, commutativity, existence of identity element and preservation of arbitrary suprema. Let* A *be a tied implication on* (L,P)*. It is always possible to find a monotone, associative tying operation* T *of* A *(Theorem 3.1). So, we need to justify the other three requirements only.*

The commutativity of T *boils down to the exchange principle for* A *(Theorem 3.1), a property satisfied in many practical examples. Moreover, both truth values conjoined by* T *act as left arguments for* A*. So, they are likely to have the same intuitive semantics. It is natural to conjoin such a pair of truth values by a commutative conjunction. There is also the practical angle of convenience. If a noncommutative* T *was allowed, its residuated implication* I *would have split into a right residuum* I_R *and a left residuum* I_L*. Accordingly, in some of its occurrences in Theorem 3.2, below,* I *would be replaced by* I_R*, and in others by* I_L*. The same splitting would take place in syntax, and we would have one more connective to deal with, causing an increase in axioms and inference rules. We prefer to postpone this added complexity, until an actual need for it materializes.*

The last two requirements, identity element and the preservation of arbitrary suprema, are necessary conditions for T *to have a residuum* I *with reasonable properties (Subsection 2.1). Also, on the practical side, these two conditions are not difficult to meet. They are satisfied perforce whenever* A *distinguishes left arguments (Theorem 3.1). Otherwise, a tying operation that satisfies all the five requirements may still be found (Example 3.1(iv)). And whenever no such operation is found, we can always create it by reducing the lattice* P*, without fundamental changes to the implication* A*. See Subsection 3.2.*

For these reasons, we take our tying operations to be supremum-preserving triangular norms.

3.2 Arguments for the two-posets approach

We expound the motivations behind the use of two posets L and P in our definition of the class ADJT of tied adjointness algebras $\Lambda = (L, P, A, K, H, T, I)$, proposed in [66] as a generalization of the one-poset approach of [1]:

- Wider scope: Our approach admits more concrete examples into ADJT, that would otherwise have been excluded, see Example 3.1(iv),(vi) and Example 6.1.

- Semantic freedom: Noncommutative conjunctions often conjoin truth values of different intuitive interpretations. Those belong to independently chosen posets, which may or may not coincide.

- Relaxation of universal bounds: Some intuitive interpretations may render the idea of a top element of L a redundant one. For instance, elements of L may be (multi-dimensional) performance scores, with no upper bound anticipated. Whereas elements of P may be possibility values, among which the top value 1 is of pivotal importance, but the possibility value 0 is of little consequence.

  Nevertheless, if P has a bottom element 0, then $\top = A(0, z)$ is the top element and $\bot = K(0, z)$ is the bottom element of L, for any $z \in L$ (Lemma 2.1(iv)). Sometimes, this value $\top$ is viewed as the infinite performance score, that cannot be achieved in reality. Reciprocally, if L has a top element $\top$ and a bottom element $\bot$, then it is easy to prove that the part $\{a \in P \mid \ a \ngeq H(\top, \bot)\}$ of P is redundant in the algebra, and can be dispensed with. Upon which, $H(\top, \bot)$ becomes the bottom element of the smaller replacement of P.

- Clarity: We found, through practice, that the two-posets approach is quite helpful in the avoidance of schemata of formulae that would do more harm than good, for instance, the schemata

  $K(K(a, y), y)$, $A(a, H(x, y))$, $H(H(x, y), z)$,

  and many others. When $L \neq P$, such expressions are illegal. And even when $L = P$, they will not participate in any indispensable universally valid inequality, except under some further assumptions, such as the associativity of the conjunction K. However, such an assumption would force L to coincide with P.

- Duality (Morsi [66]): Let $\Lambda = (L, P, A, K, H, T, I)$ be a tied adjointness algebra, and let L^{op}, the opposite poset of L, be defined by reversing the order on L. Then a tied adjointness algebra $\Lambda^{dual} =$

$\left(L^{op}, P, A^d, K^d, H^d, T, I\right)$, said to be the *dual* of Λ, is obtained by taking $A^d = K$, $K^d = A$, H^d is the symmetric of H $\left(H^d\left(x,y\right) = H\left(y,x\right)\right)$, and by keeping P, T and I unchanged. This bijection : $\Lambda \longmapsto \Lambda^{dual}$ is self-inverse. It is clear that any algebraic proof in ADJT remains a valid proof after performing this duality in all its lines. This entails that the dual of any universally valid inequality in ADJT is universally valid. This way, duality works to establish some new inferences from their duals, without new proofs. This duality principle will be made precise, within syntax, in Part 2 [69].

As this particular duality is based upon reversing the order of L and keeping the order of P, its formulation would have been impossible if L had to be indiscernible from P.

- Flexibility [66]: By allowing P to differ from L in Λ, we gain some flexibility in its handling. For instance, suppose that P is a complete lattice, that the implication A is tied by an operation D but does not distinguish left arguments, and that we cannot construct a tying operation for A with an identity element. Then we can reduce P to a smaller complete lattice $\overline{P}$ through the order-preserving retraction $: P \longrightarrow \overline{P} \subset P$ (an idempotent function onto $\overline{P}$) defined in [1, Proposition 1.6], [66] by

$$\overline{a} = \inf_{z \in L} H\left(A\left(a,z\right),z\right), \qquad a \in P. \tag{13}$$

By so doing, the restriction $A : \overline{P} \times L \longrightarrow L$ will distinguish left arguments, and will be tied by the operation T_A on $\overline{P}$ defined by $T_A\left(\overline{a}, \overline{b}\right) = \overline{D\left(a,b\right)}$, which will be a supremum-preserving triangular norm. Meanwhile, the connectives A, K, H will be unchanged basically, because we have $A\left(\overline{a}, z\right) = A\left(a,z\right)$, $K\left(\overline{a}, y\right) = K\left(a,y\right)$ for all a, y, z, while the comparator H will not be affected by this restriction, because $\overline{P}$ contains the range of H.

As an example we cite the stringent implications $A_\circ$ of Example 3.1(iv), whereby $\overline{P}$ always equals the doubleton lattice $\{0,1\}$, regardless of the original P and L.

- Feasibility: The two-posets approach in ADJT comes at no price at all. An algebraic or syntactic proof in this framework is either an exact replica of the corresponding proof in the one-poset situation, or can be omitted altogether by virtue of duality.

These seven arguments (six merit points plus the absence of demerits) constitute our rationale for adopting the two-posets approach to tied implications and conjunctions. For the record of priority, we state that this approach is due to Morsi in [66].

3.3 Properties

Theorem 3.2 *The following equivalences, identities and inequalities hold in a tied adjointness algebra* $(L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$*:* $\forall a, b, c \in P$, $\forall x, y, z, w \in L$*:*

$$\textbf{Adjointness} : \quad K(a,y) \leq_L z \quad iff \quad y \leq_L A(a,z) \quad iff \quad a \leq_P H(y,z),$$
$$\textbf{Residuation} : \quad T(a,b) \leq_P c \quad iff \quad b \leq_P I(a,c) \quad iff \quad a \leq_P I(b,c),$$

TIE 1 $H(x,x) = 1$ *(the fuzzy binary relation $H(\cdot,\cdot)$ is reflexive)*

TIE 2 $A(1,z) = z$

TIE 3 $K(1,y) = y$

TIE 4 $z \leq_L A(a,z), \quad K(a,y) \leq_L y, \quad a \leq_P H(x,x)$

TIE 5 $y \leq_L A(H(y,z),z), \quad y \leq_L A(a, K(a,y))$

TIE 6 $K(a, A(a,z)) \leq_L z, \quad K(H(y,z),y) \leq_L z$

TIE 7 $a \leq_P H(y, K(a,y)), \quad a \leq_P H(A(a,z),z)$

TIE 8 $A(a, A(b,z)) = A(b, A(a,z))$ *(the exchange principle for A)*

TIE 9 $K(a, K(b,z)) = K(b, K(a,z))$ *(the exchange principle for K)*

TIE 10 $H(x, A(a,y)) = H(K(a,x),y)$

TIE 11 $H(y,z) \leq_P H(A(a,y), A(a,z))$

TIE 12 $H(y,z) \leq_P H(K(a,y), K(a,z))$

TIE 13 $K(a, A(b,z)) \leq_L A(b, K(a,z))$

TIE 14 $K(H(z,y),x) \leq_L A(H(x,z),y)$

TIE 15 $A(T(a,b),z) = A(a, A(b,z))$ *(T ties A),*
$I(a,b) \leq_P I(T(a,c), T(b,c))$ *(T is monotone in each argument),*
$T(1,b) = b,$
$T(a, T(b,c)) = T(T(a,b), c)$ *(T is associative; that is, it ties itself),*
$T(a,b) = T(b,a)$ *(T is commutative)*

TIE 16 $K(T(a,b),z) = K(a, K(b,z))$ *(T ties K)*

TIE 17 $T(H(x,y), H(y,z)) \leq_P H(x,z)$ *(H is T-transitive)*

TIE 18 $T(H(x,y), H(w,z)) \leq_P I(H(y,w), H(x,z))$

TIE 19 $H(y,z) \leq_P I(H(x,y), H(x,z))$

TIE 20 $H(x,y) \leq_P I(H(y,z), H(x,z))$

TIE 21 $T(H(y,z), I(a,b)) \leq_P H(A(b,y), A(a,z))$

TIE 22 $T(H(y,z), I(a,b)) \leq_P H(K(a,y), K(b,z))$

TIE 23 $I(a,b) \leq_P H(A(b,z), A(a,z))$

TIE 24 $I(a,b) \leq_P H(K(a,y), K(b,y))$

TIE 25 $K(b,y) \leq_L A(a, K(T(a,b), y))$

TIE 26 $K(a, A(T(a,b), z)) \leq_L A(b,z)$

TIE 27 $T(a,b) \leq_P H(A(a, A(b,z)), z)$

TIE 28 $T(a,b) \leq_P H(y, K(a, K(b,y)))$

TIE 29 $T(b, H(y,z)) \leq_P H(A(b,y), z)$

TIE 30 $T(b, H(y,z)) \leq_P H(y, K(b,z))$

TIE 31 $K(I(a,b), y) \leq_L A(a, K(b,y))$

TIE 32 $K(a, A(b,z)) \leq_L A(I(a,b), z)$

TIE 33 $H(K(H(x,y), w), K(H(x,z), w)) \geq_P H(y,z)$ *(the algebra is* balanced*, [1], [66])*

TIE 34 $H(A(H(x,z), w), A(H(y,z), w)) \geq_P H(x,y)$

TIE 35 $A\left(H\left(A\left(a,A\left(b,y\right)\right),y\right),z\right)\leq_L A\left(a,A\left(b,z\right)\right)$

TIE 36 $K\left(H\left(y,K\left(a,K\left(b,y\right)\right)\right),z\right)\geq_L K\left(a,K\left(b,z\right)\right)$

TIE 37 $H\left(K\left(a,x\right),y\right)=I\left(a,H\left(x,y\right)\right)$

TIE 38 $H\left(K\left(a,x\right),y\right)\leq_P I(H(y,z),H(x,A\left(a,z\right)))$

TIE 39 $H\left(x,A\left(a,y\right)\right)=I\left(a,H\left(x,y\right)\right)$ *(mixed exchange principle for H,I,A).*

Proof. **Adjointness** and properties **TIE** 1-4 are parts of the definition of adjointness algebras, and **TIE** 5 -**TIE** 7 are well-known, basic properties of them, see for instance [64]. Each of properties **TIE** 8 - **TIE** 14 is equivalent to the exchange principle for A [64]. **Residuation** and the properties in **TIE** 15 are parts of the definition of tied adjointness algebras, and **TIE** 16 follows from **TIE** 15 (Theorem 3.1(i)). We prove the remaining properties.

As T ties A, $A\left(T(H(x,y),H(y,z)),z\right)=A\left(H(x,y),A(H(y,z),z)\right)$
$\geq_L A\left(H(x,y),y\right)\geq_L x$, by two applications of **TIE** 5.
This renders **TIE** 17 through **Adjointness**.
TIE 18-20 follow from **TIE** 17 by **Residuation**.
Next, $A(T\left(H\left(y,z\right),I\left(a,b\right)\right),A(a,z))$
$=A\left(T\left(T\left(H\left(y,z\right),I\left(a,b\right)\right),a\right),z\right)$ (T ties A)
$=A\left(T\left(H\left(y,z\right),T\left(I\left(a,b\right),a\right)\right),z\right)$ (T is associative)
$=A\left(T\left(H\left(y,z\right),T\left(a,I\left(a,b\right)\right)\right),z\right)$ (T is commutative)
$\geq_L A\left(T\left(H\left(y,z\right),b\right),z\right)$ (by **TIE** 6 applied to T and I)
$=A\left(T\left(b,H\left(y,z\right)\right),z\right)$ (T is commutative)
$=A\left(b,A\left(H\left(y,z\right),z\right)\right)$ (T ties A)
$\geq_L A(b,y)$, by **TIE** 5.
Now we get **TIE** 21 by applying **Adjointness** to the net inequality above.
Dually, $K\left(T\left(H\left(y,z\right),I\left(a,b\right)\right),K\left(a,y\right)\right)$
$=K\left(T\left(T\left(H\left(y,z\right),I\left(a,b\right)\right),a\right),y\right)$ (T ties K)
$=K\left(T\left(H\left(y,z\right),T\left(I\left(a,b\right),a\right)\right),y\right)$
$=K\left(T\left(H\left(y,z\right),T\left(a,I\left(a,b\right)\right)\right),y\right)$
$\leq_L K\left(T\left(H\left(y,z\right),b\right),y\right)=K\left(T\left(b,H\left(y,z\right)\right),y\right)$
$=K\left(b,K\left(H\left(y,z\right),y\right)\right)$ (T ties K)
$\leq_L K\left(b,z\right)$ (**TIE** 6).
This yields **TIE** 22 by **Adjointness**.
TIE 23 and **TIE** 24 follow from **TIE** 21 and **TIE** 22, respectively, by taking $y=z$.

As T ties K, $A\left(a,K\left(T\left(a,b\right),y\right)\right)=A\left(a,K\left(a,K\left(b,y\right)\right)\right)\geq_L K\left(b,y\right)$ (by **TIE** 5), which yields **TIE** 25. **TIE** 26 is proved similarly.

TIE 27 and **TIE** 28 follow by **Adjointness** from **TIE** 15 and **TIE** 16, respectively.

TIE 29 and **TIE** 30 follow by taking $a = 1$ in **TIE** 21 and **TIE** 22, respectively.

TIE 31 follows from **TIE** 25 by replacing b by $I(a, b)$ then applying **TIE** 6 to T and I. **TIE** 32 follows similarly from **TIE** 26.

For balance we have

$H(K(H(x, y), w), K(H(x, z), w)) \geq_P I(H(x, y), H(x, z))$ (**TIE** 24) $\geq_P H(y, z)$ (**TIE** 19), which is the inequality **TIE** 33.

TIE 33 is equivalent to each one of the three properties **TIE** 34, **TIE** 35 and **TIE** 36 [1].

For identity **TIE** 37 we have, on one hand,

$H(K(a, x), y) \leq_P I(H(x, K(a, x)), H(x, y))$ (**TIE** 19) $\leq_P I(a, H(x, y))$ (**TIE** 7). On the other hand, $I(a, H(x, y)) \leq_P H(K(a, x), K(H(x, y), x))$ (**TIE** 24) $\leq_P H(K(a, x), y)$ (**TIE** 6). These two inequalities combine in **TIE** 37.

Since H is T-transitive, $T(H(K(a,x), y), H(y, z))$

$\leq_P H(K(a, x), z) = H(x, A(a, z))$ (**TIE** 10). Hence by **Residuation** we get **TIE** 38.

Identity **TIE** 39 follows from identities **TIE** 10 and **TIE** 37.

(Note also that properties **TIE** 17, 27-37, 39 are proved already in [1].)

■

As we have mentioned in Example 3.1(i), we are entitled to consider a residuated algebra $(P, \leq_P, T, I)$ as a tied adjointness algebra $(P, \leq_P, P, \leq_P, I, T, I, T, I)$; that is, by setting $A = H = I$ and $K = T$. As such, the above 39 properties of tied adjointness algebras become algebraic properties of residuated algebras. It is interesting to notice that those will then cover all the more famous inequalities of residuated algebras. This may indicate that tied adjointness algebras constitute a particularly rich generalization of residuated algebras, which manages to retain all their desirable traits, but distributes the roles of T and I among the five operations of the algebra. Accordingly, a main objective behind tied adjointness algebras is to provide a framework, through which all useful properties of residuated algebras can extend over a much wider scope of logical connectives already in use in fuzzy logic, as demonstrated in Example 3.1.

4 Application: generalized modus ponens with successive rules

One of the most obvious applications of tied implications is in interpretations of Generalized Modus Ponens (**GMP**) inference schemata that feature successive rules. Let $\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$ be a complete tied adjointness lattice. Let U, V, W be universes, which may be finite or infinite. Let M, M^{pre} be P-valued possibility distributions (shortly, P-possibility distributions) on U. We handle them simply as modal P-fuzzy subsets of U (members of P^U that attain the value 1). Likewise, $N, N^{pre} \in P^V$ are P-possibility distributions on V, and $Q, Q^{ifr} \in L^W$ are L-possibility distributions on W. Each of the three symbols X, Y, Z denotes an unknown individual in the universe U, V or W, respectively.

A **GMP** inference scheme takes the form [90]:

Inference Scheme (I):
Rule: If X is M then Z is Q
Premise: X is M^{pre}

Inference: Z is Q^{ifr}.

We refer the reader to the comprehensive survey of fundamental contributions to **GMP**, up to 1990, by Dubois and Prade in [22, Section 4], with extensive citations.

An *interpretation* of this inference scheme is a concrete mathematical formula to compute Q^{ifr} in terms of M, Q and the observation M^{pre}, subject to a reasonable array of intuitive criteria [22]. Zadeh's *Compositional Rule of Inference* (**CRI**) [90] is the best known type of interpretations of **GMP**. Morsi [64] used in **CRI** the more general connectives of adjointness algebras. Thus, according to the combination-projection principle of Zadeh, we take:

$$Q^{ifr}(w) = \sup_{u \in U} K\left(M^{pre}(u), F(u, w)\right), \qquad w \in W. \tag{14}$$

Here, F denotes the L-possibility distribution on $U \times W$ of the rule of the scheme, which we equate to the L-fuzzy set $F \in L^{U \times W}$ given by

$$F(u, w) = A\left(M(u), Q(w)\right), \qquad (u, w) \in U \times W. \tag{15}$$

Hence, **CRI** yields the following computation of Q^{ifr}:

$$Q^{ifr}(w) = \sup_{u \in U} K\left(M^{pre}(u), A\left(M(u), Q(w)\right)\right), \qquad w \in W. \tag{16}$$

The use of a possibly noncommutative, nonassociative conjunction $K : P \times L \longrightarrow L$ is justified here, because the intuitive meaning of the P-possibility values in the observation M^{pre} may differ from that of the L-values of the rule F. Such use is a matter of necessity rather than choice. Because K has to be the adjoint of A, so that much of the intuition behind **GMP** is satisfied by this version of **CRI**, including [64] $Q^{ifr} \geq Q$, and the following criterion, due to Fukami et al. [35] and Turksen and Tian [84]:

$$Q^{ifr} = Q \quad \text{whenever} \quad M^{pre} = M. \tag{17}$$

This formulation generalizes that of Dubois and Prade [21], who used for A residuated and S-type implications on $[0, 1]$, whereas Zadeh's original formulation uses an assortment of implications on $[0, 1]$, but only min is used for K.

A generalized modus ponens scheme with successive rules takes the following form:

Inference Scheme (II):
Rule: If X is M then (if Y is N, then Z is Q)
Premise 1: X is M^{pre}
Premise 2: Y is N^{pre}

Inference: Z is Q^{ifr}.

We establish its equivalence to the following inference scheme with a compound rule, under **CRI** (16) and subject to the use of the T-conjunction to conjoin P-possibility distributions:

Inference Scheme (III):
Rule: If (X is M and Y is N) then Z is Q
Premise: X is M^{pre} and Y is N^{pre}

Inference: Z is Q^{ifr}.

We utilize the tying of A by T. But, interestingly, the commutativity of T is not needed in the proof.

Denote by G the L-possibility distribution on $V \times W$ of the rule (If Y is N then Z is Q). Then Inference Scheme (II) breaks into the following two successive inference schemata:

Inference Scheme (IIa):
Rule (a): If X is M then (Y, Z) is G
Premise 1: X is M^{pre}

Inference: (Y,Z) is G^1

followed by

Inference Scheme (IIb):
Rule (b): (Y,Z) is G^1
Premise 2: Y is N^{pre}

Inference: Z is Q^{ifr}.

Then we compute Q^{ifr} in two consecutive steps, through these two schemata. First, for $(v,w) \in V \times W$:

$$G^1(v,w) = \sup_u K\left(M^{pre}(u), A\left(M(u), G(v,w)\right)\right) \qquad \text{(by (16))}$$
$$= \sup_u K\left(M^{pre}(u), A\left(M(u), A\left(N(v), Q(w)\right)\right)\right) \qquad \text{(using (15))}$$
$$= \sup_u K\left(M^{pre}(u), A\left(T\left(M(u), N(v)\right), Q(w)\right)\right),$$

because T ties A. Second, for $w \in W$:

$$Q^{ifr}(w) = \sup_{v\in V} K\left(N^{pre}(v), G^1(v,w)\right) \qquad \text{(by (14))}$$
$$= \sup_v K\left(N^{pre}(v),\ \sup_u K\left(M^{pre}(u), A\left(T\left(M(u), N(v)\right), Q(w)\right)\right)\right)$$
$$= \sup_{u,v} K\left(N^{pre}(v),\ K\left(M^{pre}(u), A\left(T\left(M(u), N(v)\right), Q(w)\right)\right)\right),$$

by (3). As the symmetric of T ties K (Theorem 3.1), then

$$Q^{ifr}(w) = \sup_{(u,v)\in U\times V} K\left(T\left(M^{pre}(u), N^{pre}(v)\right), A\left(T\left(M(u), N(v)\right), Q(w)\right)\right). \tag{18}$$

This conclusion from Inference Scheme (II) is, clearly, also the conclusion from Inference Scheme (III), when T interprets conjunction between P-possibility distributions. This establishes their equivalence under **CRI** (16). We thank an anonymous referee for noting that one significance of the co-presence of the two conjunctions K and T in (18) is that K is used as a metalogical conjunction to conjoin premises of the inference scheme, while T is an object-level conjunction. This interpretation gives a good justification for the generalization of residuated algebras to tied adjointness ones.

As T is associative, this equivalence extends immediately to **GMP** schemata with three or more successive rules.

Inference Scheme (III), with a compound rule, is handled in some literature on **GMP** [5], [11], [39], [63], [80], [83]. The arguments here set another criterion for the intuitive acceptability of its interpretations; its equivalence to Inference Scheme (II) with a succession of simple rules. When **CRI** is used to model both schemata, we conclude that if some implication A interprets the conditional parts of all rules, then the conjunctive part in the compound rule should be interpreted by a conjunction that ties A, in order to meet this criterion.

Remark 4.1 *Starting from an approach different from the one above, Turksen and Demirli [83] established the equivalence of the inference schemata (II) and (III), in the very special case of the connectives of what is, in our terminology, the tied adjointness algebra of the Reichenbach implication on* $[0,1]$ *(the S-type implication of the t-norm Product). However, their decomposition of Scheme (II) is different from the one we proposed above, as they break it down to:*

Inference Scheme (IIc):

Rule (c): If Y *is* N *then* Z *is* Q

Premise 1: Y *is* N^{pre}

Inference: Z *is* Q^1

followed by

Inference Scheme (IId):

Rule (d): If X *is* M *then* Z *is* Q^1

Premise 2: X *is* M^{pre}

Inference: Z *is* Q^{ifr}.

Turksen and Demirli also discuss the reflections of this equivalence on computational complexity.

5 Application: many-valued rough sets

The application sketched in this section is taken from Morsi [66]. It extends to P-valued partitions and L-valued fuzzy sets some of the basic notions of the *rough sets* of Pawlak [75]. Let $\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$ be a complete tied adjointness lattice, and denote the top and bottom elements of L by 1_L and 0_L, respectively. A P-valued T*-indistinguishability relation* (or, T*-similarity relation*) on a nonempty set V is, according to Höhle [48], a function $S : V \times V \longrightarrow P$ that satisfies the following three properties: $\forall u, v, r \in V$:

Similarity1: $S(u,u) = 1$ (reflexivity).

Similarity2: $S(u,v) = S(v,u)$ (symmetry).

T-**Similarity3**: $T(S(u,v), S(v,r)) \leq S(u,r)$ (T-transitivity).

A P-valued T*-partition* of V is a subset $\mathfrak{P}$ of P^V that satisfies the following three properties:

Partition1: For every $\mu \in \mathfrak{P}$ there exists $u \in V$ such that $\mu(u) = 1$.

Partition2: For every $u \in V$ there exists a unique $\mu \in \mathfrak{P}$ such that $\mu(u) = 1$. We denote this unique μ by $[u]$.

Partition3: $\forall u, v \in V : \quad \sup_{r \in V} T\left([u](r), [v](r)\right) = [u](v)$.

This definition has been formulated in [70] within the special case $P = [0,1]$. T-partitions are shown in [70] to be in the following canonical one-to-one correspondence with T-indistinguishability relations:

$$\forall u, v \in V : \quad [u](v) = S(u, v). \tag{19}$$

As the $[0,1]$-valued T-partitions of De Baets and Mesiar [10] (formulated differently from the ones above) have exactly the same relationship to T-similarities [10, Theorem 4], they coincide with our T-partitions when $P = [0,1]$.

Throughout, $\mathfrak{P}$ is a P-valued T*-partition* of V, and S is its T-indistinguishability relation, as in (19). The *upper operator* $\mathfrak{U}$ and *lower operator* $\mathfrak{L}$, induced by $\mathfrak{P}$, are the operators on L^V defined by: $\forall \lambda \in L^V, \ \forall u \in V :$

$$\mathfrak{U}(\lambda)(u) = \sup_{r \in V} K\left([u](r), \lambda(r)\right) = \sup_{r \in V} K\left(S(u,r), \lambda(r)\right), \tag{20}$$

$$\mathfrak{L}(\lambda)(u) = \inf_{r \in V} A\left([u](r), \lambda(r)\right) = \inf_{r \in V} A\left(S(u,r), \lambda(r)\right). \tag{21}$$

These two operators generalize the corresponding ones of Morsi and Yakout [70], whereby only residuated algebras Λ on $[0,1]$ are employed. In turn, the upper operators of [70] generalize the corresponding ones of Farinas and Prade [30], whereby only the case $T = min$ is considered.

The tuple $\mathfrak{R} = (V, \mathfrak{P}, \mathfrak{U}, \mathfrak{L})$ is referred to as a (L,P)*-valued rough set*.

The following dual pairs of useful properties of $\mathfrak{R}$ follow from the properties of $\mathfrak{P}$ and of the tied adjointness algebra Λ (Theorem 3.2). Their proofs trace much the same lines of the corresponding proofs given in [70] in the case $\Lambda = ([0,1], T, I)$: For all $\lambda \in L^V$, $c \in P$ and for each constant L-fuzzy subset $\underline{z}$ of V with value $z \in L$:

1. $\mathfrak{U}(\underline{z}) = \mathfrak{L}(\underline{z}) = \underline{z}$.

2. $\mathfrak{U}$ preserves arbitrary joins and $\mathfrak{L}$ preserves arbitrary meets in L^V.

3. $\mathfrak{U}(\lambda) \geq \lambda$, and $\mathfrak{L}(\lambda) \leq \lambda$.

4. $\mathfrak{U}(\mathfrak{U}(\lambda)) = \mathfrak{U}(\lambda)$, and $\mathfrak{L}(\mathfrak{L}(\lambda)) = \mathfrak{L}(\lambda)$.

5. $\mathfrak{U}(\mathfrak{L}(\lambda)) = \mathfrak{L}(\lambda)$, and $\mathfrak{L}(\mathfrak{U}(\lambda)) = \mathfrak{U}(\lambda)$.

6. $\lambda = \mathfrak{U}(\lambda)$ if and only if $\lambda = \mathfrak{L}(\lambda)$.

7. $\mathfrak{U}(K(c,\lambda)) = K(c,\mathfrak{U}(\lambda))$, and $\mathfrak{L}(A(c,\lambda)) = A(c,\mathfrak{L}(\lambda))$.

Two coarse classifications of the L-fuzzy subsets of V are induced from $\mathfrak{P} \subsetneqq P^V$ when we replace each fuzzy set $\lambda \in L^V$ by either its *upper approximation* $\mathfrak{U}(\lambda) \in L^V$, or its *lower approximation* $\mathfrak{L}(\lambda) \in L^V$. These classifications reduce L^V to the smaller collection $\{\lambda \in L^V | \quad \lambda = \mathfrak{U}(\lambda)\} = \{\lambda \in L^V | \quad \lambda = \mathfrak{L}(\lambda)\}$. This collection becomes finite whenever both $\mathfrak{P}$ and L are finite.

Furthermore, when the implication A distinguishes left arguments, the partition $\mathfrak{P}$ is retrieved from its upper and lower operators through the formulae

$$\begin{aligned} \forall u,v &\in V: \quad [u](v) = \inf_{z\in L} H(z, \mathfrak{U}(z1_v)(u)), \\ \forall u,v &\in V: \quad [u](v) = \inf_{z\in L} H\left(\mathfrak{L}\left(\frac{z}{v}\right)(u), z\right), \end{aligned}$$

where the two L-fuzzy subsets $z1_v$ and $\frac{z}{v}$ of V are defined by:

$$z1_v(r) = \begin{cases} z, & r = v \\ 0_L, & r \neq v \end{cases}, \quad \frac{z}{v}(r) = \begin{cases} z, & r = v \\ 1_L, & r \neq v \end{cases}.$$

6 Prelinearity and representation theorem

6.1 Prelinear residuated lattices

Definition 6.1 *(Esteva and Godo [26], Hájek [45], Höhle [51]) A residuated implication I is said to be* prelinear *if it satisfies*

$$\forall a,c \in P: \qquad I(a,c) \vee I(c,a) = 1. \tag{22}$$

We also say that its residuated lattice $(P, \leq_p, \wedge, \vee, 1, T, I)$ is prelinear*.*

This terminology is justified by the observation that the condition "$x \leq y$ iff $I(x,y) = 1$" ensures that all residuated chains are prelinear. Notice that the preceding definition differs from the standard ones of Hájek and Höhle in that here we do not require the lattice to have a bottom element 0, and so we drop all mention of a negation operator. An example of a non-prelinear residuated lattice is given in [65].

Definition 6.2 *(Esteva and Godo [26]). An* **MTL**-algebra *(a monoidal t-norm-based logic algebra) is a residuated lattice with* 0*, which satisfies the prelinearity condition.*

Hájek's definition of **BL**-algebras can be rephrased as follows:

Definition 6.3 *(Hájek [45]). A* **BL**-algebra $(P, \leq_p, \wedge, \vee, 1, 0, T, I)$ *(a basic logic algebra) is a residuated lattice with* 0*, which satisfies the prelinearity condition as well as the following* divisibility *condition*

$$\forall a, c \in P : \qquad T(a, I(a, c)) = T(c, I(c, a)). \tag{23}$$

An MV-algebra *is a* ***BL****-algebra in which the negation operator:* $b \mapsto I(b, 0)$ *is involutive.*

Divisibility becomes equivalent to the continuity of T when $P = [0, 1]$ [46]. Further special cases of these types are considered in [8], [25], [26], [27], [29], [33], [41], [43], [45], [46], [47], [56].

Definition 6.4 *[26], [45] A* filter *F on a residuated lattice $(P, \leq, T, I)$ is a proper subset of P that satisfies $\forall a, b \in P$:*

(i) If $b \geq a \in F$ then $b \in F$,

(ii) If $a, b \in F$ then $T(a, b) \in F$.

A filter F is said to be prime *if $\forall a, c \in P$ we find that $I(a, c) \in F$ or $I(c, a) \in F$.*

Hájek [45] accepts P to be a filter in itself, in which case P/F becomes a singleton. Readers will notice that nothing will be lost by excluding P from the set of its filters, as we did above. (In particular, the existence of prime filters is assured by Lemma 6.4 below.) The singleton $\{1\}$ is a filter in any residuated algebra, and is contained in any other filter.

Turunen gave another characterization of filters:

Lemma 6.1 *[86] Let F be a proper subset of a residuated algebra P. Then F is a filter on P if and only if it satisfies for all a, b in P the following two conditions:*

(1) $1 \in F$,

(2) If $a \in F$ and $I(a, b) \in F$ then $b \in F$.

Proof. [86] If F is a filter and both a and $I(a, b)$ are in F, then $b \geq T(a, I(a, b)) \in F$. Conversely, assume (i) and (ii), and let a, b be in F. Then $\forall c \geq a$, $I(a, c) = 1 \in F$, and so $c \in F$. Consequently, as $I(a, T(a, b)) \geq b \in F$, then $I(a, T(a, b)) \in F$. Hence by (ii), $T(a, b) \in F$. This renders F a filter. ■

Lemma 6.2 *Let Λ be a prelinear residuated lattice, and let F be a filter on Λ. Then the following conditions are equivalent:*

(i) F is a prime filter.

(ii) for each $a, b \in P$ such that $a \vee b \in F$, $a \in F$ or $b \in F$.

(iii) for each $a, b \in P$ such that $a \vee b = 1$, $a \in F$ or $b \in F$.

Proof. *(i) entails (ii):* Assume F is a prime filter, and let $a \vee b \in F$. By prelinearity, $I(a, b) \vee I(b, a) = 1$. Since F is prime, then $I(a, b) \in F$, say, and hence $I(a \vee b, b) = I(a, b) \wedge I(b, b) = I(a, b) \in F$. Thus by Lemma 6.1, $b \in F$.

(ii) entails (iii): $1 \in F$.

(iii) entails (i): Conjoin (iii) with the equation of prelinearity (22). ■

Note that Turunen [86] adopts condition (ii) in Lemma 6.2 as a definition of prime filters on **BL**-algebras.

Definition 6.5 *A* minimal prime filter *is a prime filter that does not properly contain any other prime filter.*

When P is a chain, the singleton $\{1\}$ is the unique minimal prime filter on P, although P may possess a large number of prime filters. See also Example 6.2 below.

Lemma 6.3 *In a residuated algebra, every prime filter contains a minimal prime filter.*

Proof. Let F be a prime filter in a residuated algebra $(P, \leq, T, I)$, and denote by $\mathfrak{V}_F$ the set of all prime filters contained in F ($F \in \mathfrak{V}_F$, so it is nonempty). Let $\{F_j\}$ be a nest in $\mathfrak{V}_F$, and take $G = \bigcap_j F_j$. Then $G \neq \emptyset$ because $1 \in G$, and it is direct to verify that G is a filter contained in F. We claim that G is prime. For, otherwise, $\exists a, b \in P$ such that neither $I(a, b)$ nor $I(b, a)$ is in G. Consequently, there are two values m, k for the index j such that $I(a, b) \notin F_m$ and $I(b, a) \notin F_k$. As $\{F_j\}$ is nested, then $F_m \subseteq F_k$, say. Hence, $I(b, a) \notin F_m$ as well, and so F_m is not prime, a contradiction. This proves the claim.

We have shown that each nest in $\mathfrak{V}_F$ has a lower bound in $\mathfrak{V}_F$. So by the minimal principle (which is equivalent to the axiom of choice, see Kelley [57]), $\mathfrak{V}_F$ has minimal elements. Each of those is a minimal prime filter contained in F. ■

Lemma 6.4 *[26], [45] Let P be a prelinear residuated lattice, and let $a \in P$, $a \neq 1$. Then there is a prime filter on P not containing a.*

Corollary 6.1 *Let P be a prelinear residuated lattice, and let $a \in P$, $a \neq 1$. Then there is a minimal prime filter on P not containing a.*

6.2 Prelinear tied adjointness algebras

In a tied adjointness lattice on (L, P), the meet and join operations on P will be denoted by $\wedge$ and $\vee$, and on L by $\overline{\wedge}$ and $\underline{\vee}$.

Definition 6.6 *A* prelinear *tied adjointness algebra is a tied adjointness lattice* $\Lambda = (L, \leq_L, P, \leq_P, A, K, H, T, I, \wedge, \vee, \overline{\wedge}, \underline{\vee})$ *satisfying the prelinearity equation (22) for* I *as well as the following prelinearity equation for* H*:*

$$\forall x, y \in L : \quad H(x, y) \vee H(y, x) = 1. \tag{24}$$

We denote the class of all prelinear tied adjointness algebras by L-ADJT.

Subdirect products of tied adjointness chains are prelinear tied adjointness algebras. The converse is also true (Theorem 6.1, below).

In a prelinear tied adjointness algebra, the structure $(P, \leq_P, T, I, \wedge, \vee)$ is a *prelinear hoop* [71] (or, *basic semihoop* [28]); that is, an **MTL**-algebra (Definition 6.2) that need not have a zero element. Accordingly, all the results concerning the operations $T, I, \wedge$, and $\vee$ in **MTL**-algebras remain valid for this structure. Note that in a **BL**-algebra, both lattice operations are defined in terms of T and I. Whereas in our system, as in **MTL**-algebras, each of the join and meet operations will be obtainable, only, in terms of the other one plus the other connectives, see Lemma 6.6 and Part 2 [69].

Lemma 6.5 *A tied adjointness algebra over* (L, P) *becomes linear if and only if it is prelinear and* P *is a chain.*

Proof. Assume prelinearity and that P is a chain. As H has values in P, then, for each pair x, y in L, $H(x, y) \leq_P H(y, x)$, say, and so the prelinearity equation (24) yields $H(y, x) = 1$. Thus $y \leq_L x$. This proves that L is a chain too. The converse is immediate. ∎

Consider the following basic example in ADJT:

Example 6.1 *Let* (P, T, I) *be a complete residuated chain, let* U *be a set of two or more elements, and let* P^U *be the product lattice. Define the tied adjointness algebra* $\Lambda = \left(P^U, P, A, K, H, T, I\right)$ *as follows:* $\forall a \in P, \forall \lambda, \mu \in P^U, \forall u \in U : \quad K(a, \lambda)(u) = T(a, \lambda(u)), \quad A(a, \mu)(u) = I(a, \mu(u))$ *and* $H(\lambda, \mu) = \inf_{w \in U} I(\lambda(w), \mu(w)) \in P$*. As* P *is linear and* P^U *is not,* Λ *is not prelinear due to Lemma 6.5.*

Weaker versions of linearity and prelinearity are formulated for tied adjointness algebras; by requesting their residuated parts only to be linear, respectively prelinear (that is, only (22) is stipulated). These weak versions have been adequate for the purposes of [66], and this weak linearity accommodates the algebra in Example 6.1. Weaker versions of the results of this section should hold for these two weak notions.

Lemma 6.6 *In each prelinear tied adjointness algebra, the following hold for all a, b in P and x, y in L:*

$$x \;\veebar\; y = A(H(x,y),y) \barwedge A(H(y,x),x), \tag{25}$$
$$a \vee b \;=\; I(I(a,b),b) \wedge I(I(b,a),a) \text{ [26], [51].} \tag{26}$$

Proof. It is clear from **TIE** 4 and **TIE** 5 that $x \veebar y \leq_L A(H(x,y),y) \barwedge A(H(y,x),x)$.

On the other hand,
$A(H(x,y),y) \barwedge A(H(y,x),x)$
$\leq_L A(H(x,y), x \veebar y) \barwedge A(H(y,x), x \veebar y)$
$= A(H(x,y) \vee H(y,x), x \veebar y)$ (by (2))
$= A(1, x \veebar y)$ (by prelinearity) $= x \veebar y$. This renders (25).

Identity (26) is a well known fact from [45], [26], and follows from (25) by taking $L = P$ and $A = H = I$. ■

Lemma 6.7 *A tied adjointness lattice is prelinear if and only if it satisfies for all a, b, c in P and x, y in L the following two conditions:*

$$I(H(x,y),c) \;\leq_P\; I(I(H(y,x),c),c), \tag{27}$$
$$I(I(a,b),c) \;\leq_P\; I(I(I(b,a),c),c) \text{[26].} \tag{28}$$

Proof. *First, assume prelinearity. Then by (3),*
$T(I(H(x,y),c), I(H(y,x),c)) \leq_P I(H(x,y),c) \wedge I(H(y,x),c)$
$= I(H(x,y) \vee H(y,x), c) = I(1,c) = c.$

Inequality (27) now follows by ***Residuation****, and (28) is proved similarly.*

Conversely, by taking $c = H(x,y) \vee H(y,x)$ in (27) we get the prelinearity equation (24), and by taking $c = I(a,b) \vee I(b,a)$ in (28) we get the prelinearity equation (22). This completes the proof. ■

6.3 Representation theorem

Like **BL**-algebras and **MTL**-algebras, prelinear tied adjointness algebras will satisfy the lattice-subdirect-product representation property. In proving this, we follow a similar approach to that of Hájek [45] in **BL**-algebras.

Definition 6.7 *A* filter *on a tied adjointness algebra*

$$\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$$

is a filter on its residuated part $(P, \leq_P, T, I)$*.* Prime filters *and* minimal prime filters *on* Λ *are understood similarly.*

Lemma 6.8 *A prime filter* F *on a prelinear tied adjointness algebra satisfies:*

$$\forall x, y \in L : \qquad H(x, y) \in F \quad or \quad H(y, x) \in F. \tag{29}$$

Proof. Conjoin Lemma 6.2(iii) with the equation of prelinearity (24). ■

Definition 6.8 *Let* F *be a filter on a tied adjointness algebra* Λ*. Define two binary relations* $\approxeq_F^L$ *on* L *and* $\approxeq_F^P$ *on* P *as follows: For all* $x, y \in L$ *and* $a, b \in P$*:*

$$x \approxeq {}_F^L y \quad iff \quad (H(x, y) \in F \quad and \quad H(y, x) \in F), \tag{30}$$

$$a \approxeq {}_F^P b \quad iff \quad (I(a, b) \in F \quad and \quad I(b, a) \in F) \quad [45]. \tag{31}$$

Lemma 6.9 *The two relations* $\approxeq_F^L$ *and* $\approxeq_F^P$ *are congruencies on* Λ *(i.e., equivalence relations that preserve the two orders and all operations).*

Proof. By their definitions, these two relations are reflexive (because $1 \in F$) and symmetric. They are transitive because both H and I are T-transitive (**TIE** 17). Hence, both are equivalence relations.

The fact that $\approxeq_F^P$ is a congruency with respect to the connectives $T, I, \wedge$ and $\vee$ is known from [26], [45]. So we need only prove that the other connectives $A, K, H, \barwedge$ and $\veebar$ commute with these two binary relations. Let $y \approxeq_F^L z$ and $a \approxeq_F^P b$. Since $T(H(y, z), I(b, a)) \leq_P H(A(a, y), A(b, z))$ (**TIE** 21), then $H(A(a, y), A(b, z))$ and, similarly, $H(A(b, z), A(a, y))$ are in F; i.e., $A(a, y) \approxeq_F^L A(b, z)$. The other parts of the proof follow similarly from the inequalities of Theorem 3.2 and the two inequalities (5) and (6). ■

Definition 6.9 *Let* F *be a filter on a tied adjointness algebra*

$$\Lambda = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I).$$

The mathematical system

$$\Lambda/F = (L/F, \leq_{LF}, P/F, \leq_{PF}, A_F, K_F, H_F, T_F, \barwedge_F, \veebar_F, I_F, \wedge_F, \vee_F) \tag{32}$$

is defined by letting L/F *be the set of equivalence classes in* L *of* $\approxeq_F^L$*, letting* P/F *be the set of equivalence classes in* P *of* $\approxeq_F^P$*, and letting the nine binary operations* $A_F, K_F, H_F, T_F, I_F, \wedge_F, \vee_F, \barwedge_F, \veebar_F$ *be induced from the corresponding ones in* Λ *through quotients over the two congruencies* $\approxeq_F^L$ *and* $\approxeq_F^P$*, and by taking* $\leq_{LF}, \leq_{PF}$ *to be the two orders of the lattice operations. It is called the* quotient *of* Λ *over* F*.*

The next two assertions follow routinely from Lemma 6.9 and Lemma 6.8, respectively.

Lemma 6.10 *(Cf. [45]). The quotient Λ/F of Λ over F is a well-defined tied adjointness algebra. In particular, F is the top element of $(P/F, \leq_{PF})$, and the partial orders $\leq_{LF}$ and $\leq_{PF}$ are obtainable through quotient, as follows:*

$$\forall x, y \in L : \quad [x]^L_F \leq_{LF} [y]^L_F \text{ iff } H(x,y) \in F \text{ iff } H_F\left([x]^L_F, [y]^L_F\right) = F, \quad (33)$$
$$\forall a, b \in P : \quad [a]^P_F \leq_{PF} [b]^P_F \text{ iff } I(a,b) \in F \text{ iff } I_F\left([a]^P_F, [b]^P_F\right) = F. \quad (34)$$

Lemma 6.11 *(Cf. [45]). When Λ is prelinear, the quotient Λ/F is prelinear. It becomes linear if and only if F is prime.*

The definition of subdirect products of algebraic systems can be looked up in Hájek's book [45].

Theorem 6.1 (Representation Theorem) *Each prelinear tied adjointness algebra is a subdirect product of a system of tied adjointness chains.*

Proof. (Cf. Hájek [45].) Let Ω be the system of all minimal prime filters on a prelinear tied adjointness algebra $\Lambda = (L, \leq_L, P, \leq_P, A, K, H, T, I, \wedge, \vee, \overline{\wedge}, \underline{\vee})$. For $F \in \Omega$ let $\Lambda_F = \Lambda/F$ and let

$$\Lambda^\star = \prod_{F \in \Omega} \Lambda_F$$

be the direct product of the tied adjointness chains $\{\Lambda_F : F \in \Omega\}$. For $x \in L$ let $i(x)$ be the element $\left([x]^L_F\right)_{F\in\Omega}$ of $L^\star$. For $a \in P$ let $j(a)$ be the element $\left([a]^P_F\right)_{F\in\Omega}$ of $P^\star$. Together, these two functions preserve all operations. It remains to show that each of them is one to one. If $x, y \in L$ and $x \neq y$ then $x \nleq_L y$, say. Then $H(x,y) \neq 1$ in P. By Corollary 6.1, there exists a minimal prime filter F on Λ not containing $H(x,y)$. Then in Λ/F, $[x]^L_F \nleq_{LF} [y]^L_F$, hence $[x]^L_F \neq [y]^L_F$, and therefore $i(x) \neq i(y)$. If $a, b \in P$ are distinct, then similarly there exists a minimal prime filter F on Λ such that in Λ/F, $[a]^P_F \neq [b]^P_F$, and therefore $j(a) \neq j(b)$. ■

Corollary 6.2 *Each formula which is a tautology for all tied adjointness chains is a tautology for all prelinear tied adjointness algebras.*

Hájek [45] uses all prime filters in his subdirect product representation for **BL**-algebras, rather than minimal prime filters only. The problem with this approach is that, in concrete examples, we may have too many prime filters; leading to a rather cumbersome representation. This is illustrated in the following example.

Example 6.2 *Consider a product $\Lambda = (L_1, P_1) \times \cdots \times (L_n, P_n)$ of n tied adjointness chains with top elements $1_1 \in P_1, \cdots, 1_n \in P_n$, respectively (all operations are defined coordinate-wise). It is easy to see that Λ has exactly n minimal prime filters, namely*

$F_j = P_1 \times \cdots \times \{1_j\} \times \cdots \times P_n, \qquad j = 1, \cdots, n.$

However, there may be a large (sometimes infinite) number of prime filters on that product; depending on the triangular norms used.

It is direct to verify that each quotient Λ/F_j is naturally isomorphic to the tied adjointness chain (L_j, P_j). Accordingly, the subdirect product representation of this prelinear tied adjointness algebra, in the manner of Theorem 6.1, is essentially the identity isomorphism from Λ onto itself. But, if we had used all prime filters in the representation, rather than just the minimal ones, the resulting representation could have been an embedding into a much larger product of tied adjointness chains.

7 Conclusion

We studied the class ADJT of tied adjointness algebras (which are five-connectives algebras), in which the implications (equivalently, the conjunctions) are tied in the sense of Abdel-Hamid and Morsi [1], and satisfy the exchange principle. Following Morsi [66], we have generalized this notion a bit, by allowing for two partially ordered sets in each algebra, and we argued for the merits of this generalization. The class ADJT is much wider than the class of residuated algebras, and exhibits more implications employed frequently in fuzzy logic. Nevertheless, we have managed to show that all the algebraic inequalities we know to be valid in all residuated algebras remain true for tied implications and conjunctions, but in forms that assign roles for the five connectives of the algebra.

We illustrated the potential of our theory, through two applications to fuzzy logic. In one of them, we set an intuitive criterion on interpretations of a **GMP** inference scheme with a compound rule; namely, its equivalence to an inference scheme with a succession of simple rules. We showed that this criterion is met, under **CRI**, whenever an implication A interprets the conditional parts of all rules, and a conjunction that ties A interprets the conjunctive part in the compound rule. The other application, quoted from [66], was the formulation and study of a notion of (L, P)-valued rough sets, using the connectives of a tied adjointness algebra. Those obey some of the behaviour of the rough sets of Pawlak [75].

We then introduced prelinear tied adjointness algebras, in which two comparators (H and I) have been prelinear. We provided a representation of those algebras, as subdirect products of tied adjointness chains, on the lines of Hájek's representation of **BL**-algebras. But our representations have been more economical, because we utilized minimal prime filters (on residuated lattices) only; rather than all prime filters.

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