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THE BURR X FRÉCHET DISTRIBUTION WITH ITS PROPERTIES AND APPLICATIONS

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SUMMARY

In this paper we define and study a new three-parameter lifetime model called the Burr X Fréchet distribution. The new model has the advantage of being capable of modeling various shapes of aging and failure criteria. Various of its properties including ordinary and incomplete moments, quantile and generating functions, Rényi and η-entropies and order statistics are derived. The maximum likelihood method is used to estimate the model parameters. Simulation results to assess the performance of the maximum likelihood estimation are discussed. We prove empirically the importance and flexibility of the new model comparing it with other extensions of the Fréchet distribution in the existing literature.

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1 Introduction

Recently, the statisticians have proposed hundreds of continuous univariate distributions which have several applications from economics, biomedical sciences, finance and engineering, among others. These applications have shown that data sets following the classical distributions are more often the exception rather than the reality. So, a significant progress has been made towards the generalization of some well-known models and their successful applications in several applied areas. The generalization of the classical distributions is made by adding one or more shape parameter(s) to the existing probability distribution to improve the flexibility and goodness of fits of the distribution against the intuition of model parsimony.

The Fréchet (Fr) distribution (type II extreme value distribution) is one of the important distributions in extreme value theory and it has wide applicability in extreme value theory. The Fr distribution was proposed by Maurice Fréchet (1878-1973), who investigated it as one possible limit distribution for a sequence of maxima. The Fr distribution is widely used in applications involving stochastic phenomena such as rainfall, floods, air pollution as shown by Kotz and Nadarajah (2000). Harlow (2002) applied it in material properties in engineering applications. Zaharim et al. (2009) used the Fr model in analyzing wind speed data. Resnick (2013) applied the Fr distribution on point processes and regularly varying functions. For more details about the Fr distribution and its applications see, e.g. Kotz and Nadarajah (2000).

In this paper, we introduce and study a new lifetime model called the *Burr X Fréchet* (BrXFr) distribution. Using the Burr-X generator (BrX-G) introduced by Yousof et al. (2016), we construct the three-parameter BrXFr model and derive some of its mathematical properties.

In fact, the BrXFr distribution can provide better fits than at least thirteen other models, each one having the same or more number of parameters. Furthermore, the BrXFr distribution due to its flexibility in accommodating all forms of the hazard rate function (hrf) seems to be an important distribution that can be used to serve as an alternative model to other lifetime distributions available in the literature for modeling positive real data in many areas. We prove that the BrXFr distribution is capable of modelling various shapes of data.

Let $g(x;\xi)$ and $G(x;\xi)$ denote the probability density function (pdf) and the cumulative distribution function (cdf) of the baseline model with parameter vector ξ, respectively. Hence, the cdf of the BrX-G is defined by

$$F(x;\theta,\xi)=\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta}. \tag{1.1}$$

The corresponding pdf of the BrX-G is given by

$$f(x;\theta,\xi)=\frac{2\theta g(x;\xi)G(x;\xi)}{\overline{G}(x;\xi)^3}\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta-1}, \tag{1.2}$$

where θ is a positive shape parameter and $\overline{G}(x;\xi) = 1 - G(x;\xi)$.

The hrf and cumulative hazard rate function (chrf) of the BrX-G are given, respectively, by

$$h(x;\theta,\xi) = \frac{2\theta g(x;\xi)G(x;\xi)\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta-1}}{\overline{G}(x;\xi)^3\left(1-\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta}\right)}$$

and

$$H(x;\theta,\xi) = -\left[\log\left(1-\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta}\right)\right].$$

In general a random variable X with pdf (1.2) is denoted by $X \sim$BrX-G(θ,ξ).

2 Existing literature

In this section, we shall survey all the extensions of the Fr distribution and considered competitive models to the proposed distribution. The statistical literature contains many modified forms for the Fr distributions which are mentioned below in chronological order. We prove empirically that the BrXFr model can give better fits over all other extensions of the Fr distribution. We have the following models:

The exponentiated Fr (EFr) (Nadarajah and Kotz, 2003). The pdf of EFr is defined as

$$f(x){=}\theta\beta\alpha^{\beta}\,x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\,\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{\theta-1},$$

where $\alpha,\beta,\theta > 0$.

The Beta Fr (BFr) (Nadarajah and Gupta, 2004). The pdf of BFr is defined as

$$f(x){=}\frac{\beta\alpha^{\beta}}{B\left(a,b\right)}\,x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\,\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1},$$

where $\alpha,\beta,a,b > 0$.

The transmuted Fr (TFr) (Mahmoud and Mandouh, 2013). The pdf of TFr is defined as

$$f(x){=}\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{\theta+1-2\theta\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\},$$

where $\alpha,\beta > 0$ and $|\theta| \leq 1$.

The Marshall-Olkin Fr (MOFr) (Krishna et al., 2013). The pdf of MOFr is defined as

$$f(x){=}\frac{\theta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\left\{\theta+(1-\theta)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{2}},$$

where $\alpha, \beta, \theta > 0$.

The gamma extended Fr (GEFr) (Silva el al., 2013). The pdf of GEFr is defined as

$$\begin{aligned} f(x) &= \frac{a\beta\alpha^{\beta}}{\Gamma(b)} x^{-\beta-1} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1} \\ &\quad \times \left\{-\log\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right\}^{b-1}, \end{aligned}$$

where $\alpha, \beta, a, b > 0$.

The transmuted exponentiated Fr (TEFr) (Elbatal et al., 2014). The pdf of TEFr is defined as

$$\begin{aligned} f(x) &= a\beta\alpha^{\beta} x^{-\beta-1} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1} \\ &\quad \times \left(1-b+2b\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right), \end{aligned}$$

where $\alpha, \beta, a > 0$ and $|b| \leq 1$.

The Kumaraswamy Fr (KFr) (Mead and Abd-Eltawab, 2014). The pdf of KFr is defined as

$$f(x)=ab\beta\alpha^{\beta} x^{-\beta-1} \exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] \left\{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1},$$

where $\alpha, \beta, a, b > 0$.

The transmuted Marshall-Olkin Fr (TMOFr) (Afify et al., 2015). The pdf of TMOFr is defined as

$$f(x)=\frac{a\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\left\{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{2}} \left\{1+b-\frac{2b\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\},$$

where $\alpha, \beta, a > 0$ and $|b| \leq 1$.

The transmuted exponentiated generalized Fr (TEGFr) (Yousof et al., 2015). The pdf of TEGFr is defined as

$$\begin{aligned} f(x) &= \frac{ab\beta\alpha^{\beta}}{x^{\beta+1}} \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1} \left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right)^{b-1} \\ &\quad \times\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \left\{1+\delta-2\delta\left\{1-\left[1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right]^{a}\right\}^{b}\right\}, \end{aligned}$$

where $\alpha, \beta, a, b > 0$ and $|\delta| \leq 1$.

The Kumaraswamy Marshall-Olkin Fr (KMOFr) (Afify et al. 2016a). The pdf of KMOFr is defined as

$$
\begin{aligned}
f(x) &= ab\delta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{\delta+(1-\delta)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-a-1} \\
&\quad\times\left\{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{\delta+(1-\delta)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-a}\right\}^{b-1},
\end{aligned}
$$

where $\alpha, \beta, a, b, \delta > 0$.

The Kumaraswamy transmuted Marshall-Olkin Fr (KTMOFr) (Yousof et al., 2016). The pdf of KTMOFr is defined as

$$
\begin{aligned}
f(x) &= \frac{ab\delta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{[\delta+K(x)]^{2}}\left\{1+\lambda-\frac{2\lambda\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\delta+K(x)}\right\} \\
&\quad\times\left\{\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\delta+K(x)}\left[1+\lambda-\frac{\lambda\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\delta+K(x)}\right]\right\}^{a-1} \\
&\quad\times\left(1-\left\{\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\delta+K(x)}\left[1+\lambda-\frac{\lambda\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\delta+K(x)}\right]\right\}^{a}\right)^{b-1},
\end{aligned}
$$

where $K(x) = (1-\delta)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$, $\alpha, \beta, a, b, \delta > 0$ and $|\lambda| \leq 1$.

The Weibull Fr (WFr) (Afify et al., 2016b). The pdf of WFr is defined as

$$
\begin{aligned}
f(x) &= ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-b-1} \\
&\quad\times\exp\left(-a\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]-1\right\}^{-b}\right),
\end{aligned}
$$

where $\alpha, \beta, a, b > 0$.

The exponentiated exponential Fr (EExFr) (Mansoor et al., 2016). The pdf of EExFr is defined as

$$
\begin{aligned}
f(x) &= ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1} \\
&\quad\times\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b}\right)^{a-1},
\end{aligned}
$$

where $\alpha, \beta, a, b > 0$.

The beta exponential Fr (BExFr) (Mead et al., 2017). The pdf of BExFr is defined as

$$\begin{aligned} f(x) &= \frac{\delta\beta\alpha^{\beta}}{B(a,b)} x^{-\beta-1} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{\delta b-1} \\ &\quad \times \left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{\delta}\right)^{a-1}, \end{aligned}$$

where $\alpha, \beta, a, b, \delta > 0$.

The modified Fr (MFr) (Tablada and Cordeiro, 2017). The pdf of MFr is defined as

$$f(x)=\frac{1}{x}(\beta+\theta x)\left(\frac{\alpha}{x}\right)^{\beta} \exp\left[-\theta x-\left(\frac{\alpha}{x}\right)^{\beta} \exp(-\theta x)\right],$$

where $\alpha, \beta, \theta > 0$.

All the above models will be used in the empirical comparisons in the application section except the KTMOFr distribution.

The rest of the paper is organized as follows. In Section 3, we define the BrXFr distribution and provide some plots for its pdf and hrf. In Section 4, we provide a useful mixture representation for its pdf. In Section 5, we derive some mathematical properties of the BrXFr distribution including ordinary and incomplete moments, quantile and generating functions, Galtonś skewness and Moors´ kurtosis, moments of the residual and reversed residual life, Rényi and η-entropies and order statistics. The maximum likelihood estimation of the unknown model parameters is addressed in Section 6. In Section 7, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 8, we provide an application to a real data set to illustrate the importance and flexibility of the new model. Finally, in Section 9, we give some concluding remarks.

3 The BrXFr distribution

The cdf and pdf of the Fr distribution are given, respectively, by $G(x;\alpha,\beta) = \exp[-\left(\frac{\alpha}{x}\right)^{\beta}]$ and $g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{-\beta-1}\exp[-\left(\frac{\alpha}{x}\right)^{\beta}]$, where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter. Based on Equation (1.1), the cdf of the BrXFr distribution is defined (for $x > 0$) by

$$F(x)=\left[1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]\right]^{\theta}. \qquad (3.1)$$

Using Equation (1.2), we have the pdf of the BrXFr distribution

$$f(x) = \frac{2\theta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-2\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{3}}\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]$$
$$\times\left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]\right\}^{\theta-1}. \tag{3.2}$$

Henceforth, let $X\sim$BrXFr(α,β,θ) be a random variable having the pdf (3.2). The hrf and chrf of X are given, respectively, by

$$h(x) = \frac{2\theta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-2\left(\frac{\alpha}{x}\right)^{\beta}\right]\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{3}\left(1-\left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]\right\}^{\theta}\right)}$$
$$\times\left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]\right\},$$

and

$$H(x) = -\log\left(1-\left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right]\right\}^{\theta}\right).$$

Figure 1 displays some plots of the BrXFr density for selected values of α, β and θ. These plots illustrate the versatility and modality of this distribution. The plots in Figure 2 reveal that the hrf of BrXFr distribution can have bathtub, unimodal, increasing, decreasing and modified bathtub shapes.

4 Linear representation

An expansion for equation (3.1) can be derived using the power series

$$(1-z)^{b} = \sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}z^{j}, \tag{4.1}$$

where $|z|<1$ and $b>0$.

Then, the BrXFr cdf can be expressed as

$$F(x)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\theta}{j}\exp\left[-j\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right)^{2}\right].$$

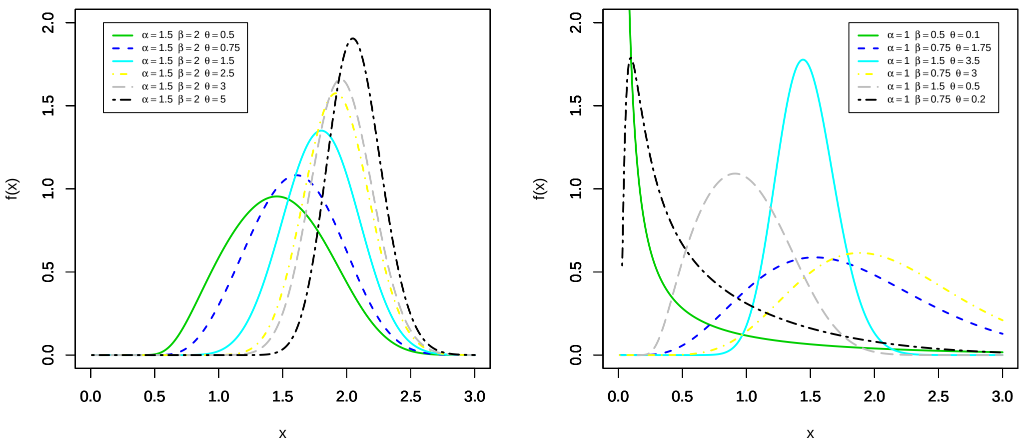


Figure 1: The BrXFr density for some selected parameter values.

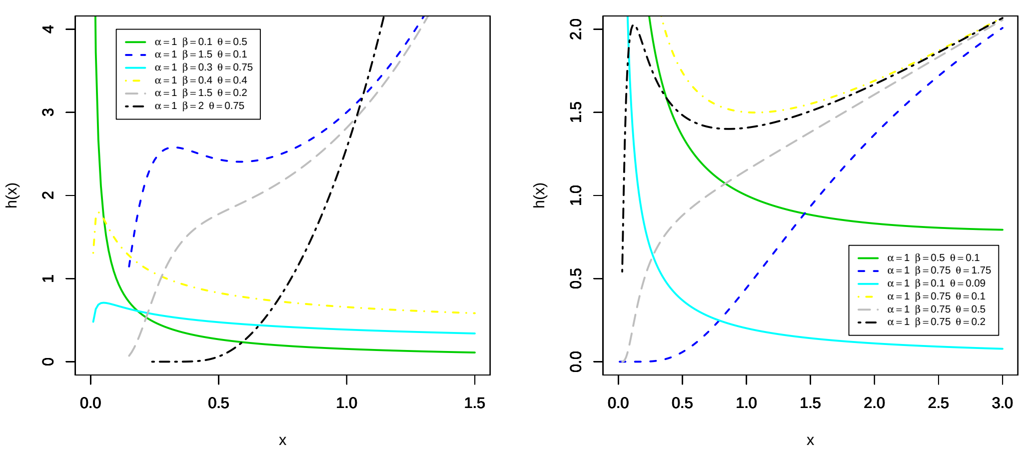


Figure 2: The BrXFr hrf for some selected parameter values.

Applying the exponential series, we have

$$F(x)=\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}j^{k}}{k!}\binom{\theta}{j}\exp\left[-2k\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2k}. \tag{4.2}$$

For $|z|<1$, the power series holds

$$(1-z)^{-q}=\sum_{j=0}^{\infty}(-1)^{j}\binom{-q}{j}z^{j}. \tag{4.3}$$

Applying the power series (4.3) to $\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2k}$ gives

$$\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2k}=\sum_{l=0}^{\infty}(-1)^{l}\binom{-2k}{l}\exp\left[-l\left(\frac{\alpha}{x}\right)^{\beta}\right].$$

Substituting the last expression in (4.2), the BrXFr cdf be expressed as

$$F(x)=\sum_{j,k,l=0}^{\infty}\frac{(-1)^{j+k+l}j^{k}}{k!}\binom{\theta}{j}\binom{-2k}{l}\exp\left[-(2k+l)\left(\frac{\alpha}{x}\right)^{\beta}\right].$$

Or, equivalently, we can write

$$F(x)=\sum_{k,l=0}^{\infty}\varphi_{k,l}\;G_{2k+l}(x), \tag{4.4}$$

where

$$\varphi_{k,l}=\sum_{j=0}^{\infty}\frac{(-1)^{j+k+l}j^{k}}{k!}\binom{\theta}{j}\binom{-2k}{l}$$

and $G_{2k+l}(x)$ is the cdf of the Fr distribution with shape parameter β and scale parameter $\alpha\,(2k+l)^{1/\beta}$.

By differentiating equation (4.4), the BrXFr density function can be expressed as a linear combination of Fr densities

$$f(x)=\sum_{k,l=0}^{\infty}\varphi_{k,l}\;g_{2k+l}(x), \tag{4.5}$$

where $g_{2k+l}(x)$ is the Fr density with shape parameter β and scale parameter $\alpha\,(2k+l)^{1/\beta}$. Hence, several mathematical properties of the BrXFr distribution can be derived from (4.5). Let the random variable Z be a Fr density with pdf and cdf defined in the beginning of Section 2. For $n<\beta$, the nth ordinary and incomplete moments of Z are given by

$$\mu_n'=\alpha^{n}\,\Gamma\left(1-\frac{n}{\beta}\right)\qquad\text{and}\qquad\varphi_{n,Z}(t)=\alpha^{n}\;\gamma\left(1-\frac{n}{\beta},\left(\frac{\alpha}{t}\right)^{\beta}\right),$$

respectively, where $\Gamma(s)=\int_0^{\infty}y^{s-1}\,e^{-y}dy$ is the complete gamma function and $\gamma(s,z)=\int_0^{z}y^{s-1}\,e^{-y}dy$ is the lower incomplete gamma function.

5 Mathematical properties

In this section, we investigate some mathematical properties of the BrXFr distribution including ordinary and incomplete moments, quantile and generating functions, Galtonś skewness and Moors´ kurtosis, moments of the residual and reversed residual life, Rényi and η-entropies and order statistics.

5.1 Ordinary and incomplete moments

The rth ordinary moment of X is given by

$$\mu'_r = E(X^r) = \sum_{k,l=0}^{\infty} \varphi_{k,l} \int_0^{\infty} x^r \, g_{2k+l}(x)dx.$$

For $r < \beta$, we obtain

$$\mu'_r = \alpha^r \, \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{k,l=0}^{\infty} \varphi_{k,l} \, (2k+l)^{r/\beta} \, . \tag{5.1}$$

The mean of X follows directly from (5.1) with $r = 1$.

The nth central moment of X, say μ_n, follows as

$$\mu_n = E(X-\mu)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu_1'^k \, \mu'_{n-k}.$$

Table 1 provides numerical values for the mean, median and standard deviation (SD) of X for selected parameter values. For fixed values of α and β, the mean and the median of BrXFr increase as θ increases, while its SD decreases as θ increases. For fixed values of β and θ, the mean, the median and the SD of BrXFr increase as α increases. For fixed values of α and θ, the SD of BrXFr decreases as β increases, while the mean and the median of BrXFr first increase and then decrease as β increases.

The skewness and kurtosis measures can be evaluated from the ordinary moments using well-known relationships, and since α in (5.1) just depends on r, then the skewness and kurtosis measures are not depend on α. Table 2 provides numerical values for the skewness and kurtosis of X for selected parameter values to illustrate their effects on these measures. For fixed value of β, the skewness of BrXFr first decrease and then increase as θ increases. For fixed value of θ, the skewness of BrXFr first decrease and then increase as θ increases.

The rth incomplete moment, say $w_r(t)$, of the BrXFr distribution is given by $w_r(t) = \int_0^t x^r \, f(x)dx$.

We can write from equation (4.5)

$$w_r(t) = \sum_{k,l=0}^{\infty} \varphi_{k,l} \int_0^{t} x^r \, g_{2k+l}(x)dx.$$

Then, we have (for $r < \beta$),

$$w_r(t) = \alpha^r \sum_{k,l=0}^{\infty} \varphi_{k,l}\ (2k+l)^{r/\beta}\ \gamma\left(1-\frac{r}{\beta}, (2k+l)\left(\frac{\alpha}{t}\right)^{\beta}\right).$$

The first incomplete moment, $w_1(t)$, has an important application related to the Bonferroni and Lorenz curves defined by $L(p) = w_1(x_p)/\mu_1'$ and $B(p) = w_1(x_p)/(p\mu_1')$, respectively, where x_p is the quantile function (qf) of X and $\mu_1' = E(X)$ is the mean of X. These curves are very useful in economics, demography, insurance, engineering and medicine.

Further, the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean and about the median of X (say say δ_μ and δ_M respectively) can be expressed as $\delta_\mu = \int_0^\infty |X - \mu_1'|\, f(x)dx = 2\mu_1'\, F(\mu_1') - 2w_1\,(\mu_1')$ and $\delta_M = \int_0^\infty |X - M|\, f(x)dx = \mu_1' - 2w_1(M)$, respectively, where $F(\mu_1')$ is evaluated from (3.1) and $w_1\,(\mu_1')$ is the first incomplete moment of X at μ_1'.

Table 1: Mean, median and SD for selected parameter values

Parameters		$\alpha = 0.7$			$\alpha = 1.0$			$\alpha = 3.0$		
θ	β	Mean	Median	SD	Mean	Median	SD	Mean	Median	SD
0.1		0.54941	0.54498	0.10220	0.77813	0.77854	0.14522	2.35474	2.33561	0.43854
1.0		0.73057	0.73398	0.05732	1.04367	1.04855	0.08188	3.13102	3.14564	0.24565
2.5	5.0	0.77167	0.77284	0.04075	1.10238	1.10405	0.05822	3.30715	3.31216	0.17465
4.0		0.78785	0.78819	0.03480	1.12550	1.12599	0.04972	3.37649	3.37798	0.14915
6.0		0.79985	0.79968	0.03075	1.14265	1.14240	0.04393	3.42794	3.42721	0.13179
10.0		0.81296	0.81234	0.02673	1.16138	1.16049	0.03819	3.48413	3.48148	0.11457
0.1		0.56837	0.56819	0.08790	0.80160	0.81171	0.12603	2.43163	2.43512	0.37799
1.0		0.72507	0.72821	0.04755	1.03581	1.04030	0.06792	3.10744	3.12089	0.20377
2.5	6.0	0.75908	0.76019	0.03345	1.08441	1.08599	0.04778	3.25322	3.25797	0.14335
4.0		0.77237	0.77276	0.02845	1.10339	1.10394	0.04064	3.31016	3.31183	0.12192
6.0		0.78219	0.78213	0.02506	1.11742	1.11733	0.03581	3.35225	3.35200	0.10742
10.0		0.79288	0.79244	0.02172	1.13269	1.13206	0.03103	3.39807	3.39617	0.09310
0.1		0.52302	0.59861	0.08701	0.76244	0.85516	0.12039	2.32547	2.56549	0.33615
1.0		0.71843	0.72105	0.03547	1.02632	1.03007	0.05067	3.07897	3.09022	0.15201
2.5	8.0	0.74373	0.74468	0.02462	1.06247	1.06382	0.03517	3.18740	3.19147	0.10550
4.0		0.75351	0.75389	0.02083	1.07644	1.07698	0.02976	3.22933	3.23095	0.08927
6.0		0.76071	0.76074	0.01829	1.08672	1.08677	0.02612	3.26017	3.26031	0.07837
10.0		0.76851	0.76824	0.01579	1.09788	1.09749	0.02256	3.29363	3.29248	0.06767

Table 2: Skewness and Kurtosis for selected parameter values

θ	β	Sk	Ku	β	Sk	Ku	β	Sk	Ku
0.5		-0.22673	2.59646		-0.26947	2.63078		1.27616	1.65362
1.0		-0.29013	2.89619		-0.32673	2.93625		-0.37302	2.99329
2.5	5.0	-0.16545	3.00926	6.0	-0.19198	3.02882	8.0	-0.22535	3.05736
4.0		-0.06410	3.00607		-0.08635	3.01430		-0.11425	3.02709
6.0		0.02356	3.01080		0.00422	3.01133		-0.02000	3.01393
10.0		0.12612	3.03789		0.10955	3.03145		0.08885	3.02504

5.2 Quantile and generating functions

The quantile function (qf) of the BrXFr, $Q(.)$, is giving by

$$Q(u) = \alpha \left[-\log \left(\frac{\left[-\log \left(1 - u^{\frac{1}{\theta}} \right) \right]^{\frac{1}{2}}}{1 + \left[-\log \left(1 - u^{\frac{1}{\theta}} \right) \right]^{\frac{1}{2}}} \right) \right]^{-\frac{1}{\beta}}, \ 0 < u < 1. \tag{5.2}$$

Simulating the BrXFr random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ has pdf (3.2). Furthermore, The median of the random variable $X \sim$BrXFr(α,β,θ) is a special case from (5.2) when $u = 0.5$, i.e, $M = Q(0.5)$.

To derive the the moment generating function (mgf) of the random variable $X \sim BrXFr(\alpha,\beta,\theta)$, at first, we provide the mgf of Fr distribution as given in Afify et al.(2016b). Let the random variable $Z \sim$Fr(α,β), and let $y = z^{-1}$, then the mgf of the Fr model, $M(t;\alpha,\beta)$, comes out as

$$M(t;\alpha,\beta) = \beta\alpha^{\beta} \int_0^{\infty} \exp\left(\frac{t}{y}\right) y^{\beta-1} \exp\left[-(\alpha y)^{\beta}\right] dy.$$

Applying the exponential series for $\exp(t/y)$ and after some algebra, we obtain

$$\begin{aligned} M(t;\alpha,\beta) &= \beta\alpha^{\beta} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} y^{\beta-n-1} \exp\left[-(\alpha y)^{\beta}\right] dy \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} \Gamma\left(\frac{\beta-n}{\beta}\right). \end{aligned}$$

Consider the Wright generalized hypergeometric function given by

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{array}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(\alpha_j + A_j\, n\right)}{\prod_{j=1}^{q} \Gamma\left(\beta_j + B_j\, n\right)} \frac{x^n}{n!}.
$$

Then, Afify et al. (2016b) derived the mgf of the Fr distribution as

$$
M(t;\alpha,\beta) = {}_1\Psi_0\left[\begin{array}{c}\left(1, -\beta^{-1}\right)\\ -\end{array}; \alpha\, t\right]. \tag{5.3}
$$

Using equations (4.5) and (5.3), the mgf of X ∼BrXFr(α, β, θ), denoted by $M(t)$, is given by

$$
M(t) = \sum_{k,l=0}^{\infty} \varphi_{k,l}\, {}_1\Psi_0\left[\begin{array}{c}\left(1, -\beta^{-1}\right)\\ -\end{array}; \alpha t\ (2k+l)^{1/\beta}\right].
$$

5.3 Skewness and kurtosis based on quantiles

The qf in (5.2) can be used to study the relationships between the parameters β, θ and the skewness and kurtosis, the Galtonś skewness (Galton, 1883) and Moors´ kurtosis (Moors, 1988) are depend on the qf $Q(.)$ in (5.2). The Galtonś skewness and Moors´ kurtosis, respectively, are given by

$$
S = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}
$$

and

$$
K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \tag{5.4}
$$

The parameter α can be canceled out from the S and K in (5.4), so the Galtonś skewness and the Moors´ kurtosis for the BrXFr does not depend on α.

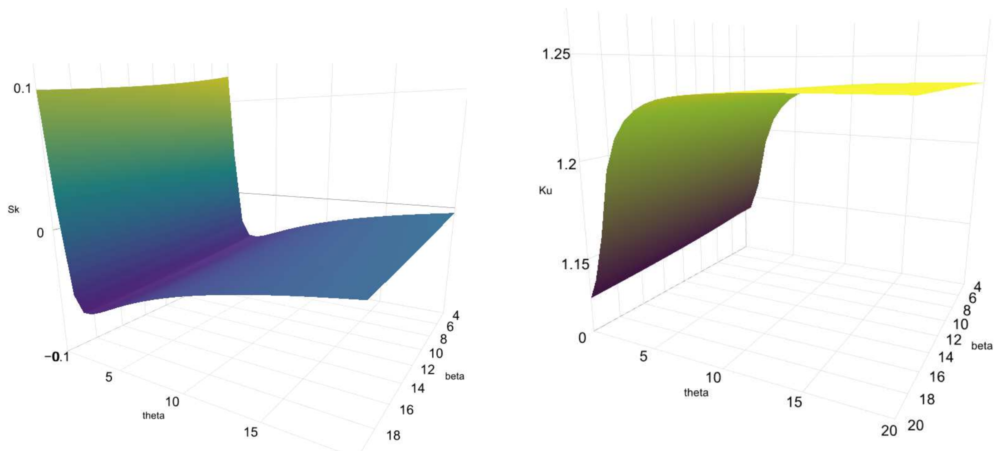
Figure 3 shows the Galtonś skewness and the Moors´ kurtosis for the BrXFr using the parameters β and θ.

5.4 Residual and reversed residual life

Let X be a random variable representing the life length for a certain unit at age t (where this unit can have multiple interpretations). Then, the random variable $X_t = X - t \mid X > t$ denotes the remaining lifetime beyond that age.

The nth moment of the residual life of X is defined (for $t > 0$ and $n = 1, 2, \ldots$) by

$$
m_n(t) = E[(X-t)^n \mid X > t] = \frac{1}{S(t)} \int_t^{\infty} (x-t)^n f(x) dx.
$$



(a) Galtonś skewness

(b) Moors´ kurtosis

Figure 3: Galtonś skewness and Moors´ kurtosis for the BrXFr distribution.

For the BrXFr distribution, we can write (for $r < \beta$)

$$m_n(t) \quad = \quad \frac{1}{S(t)} \sum_{r=0}^{n} \sum_{k,l=0}^{\infty} \varphi_{k,l} (-1)^{n-r} t^{n-r} \binom{n}{r} \alpha^r \, (2k+l)^{r/\beta} \, \Gamma\left(1 - \frac{r}{\beta}, (2k+l) \left(\frac{\alpha}{t}\right)^{\beta}\right),$$

where $\Gamma(s,z) = \int_z^{\infty} y^{s-1} \, \mathrm{e}^{-y} dy$ is the the upper incomplete gamma function.

The mean residual life (MRL) function corresponding to $m_n(t)$ represents the expected additional life length for a unit that is alive at age x and it has applications in survival analysis, biomedical sciences, life insurance, maintenance and product quality control, economics, social studies and demography.

Furthermore, Navarro et al. (1998) proved that the nth moment of the reversed residual life uniquely determines $F(x)$ and it is given (for $t > 0$ and $n = 1, 2, \ldots$) by

$$M_n(t) = E\left[(t - X)^n \mid X \le t\right] = \frac{1}{F(t)} \int_0^t (t-x)^n f(x) dx.$$

Then, the nth moment of the reversed residual life for the BrXFr distribution reduces (for $r < \beta$) to

$$M_n(t) \quad = \quad \frac{1}{F(t)} \sum_{r=0}^{n} \sum_{k,l=0}^{\infty} \varphi_{k,l} \; (-1)^r t^{n-r} \binom{n}{r} \alpha^r \, (2k+l)^{r/\beta} \, \gamma \left(1 - \frac{r}{\beta}, (2k+l) \left(\frac{\alpha}{t}\right)^{\beta}\right).$$

The mean reversed residual life (MRRL) function corresponding to $M_1(t)$ represents the waiting time elapsed for the failure of an item under the condition that this failure had occurred in $(0,t)$. It is also known as mean inactivity time (MIT). The MIT of X can be obtained by setting $n=1$ in the last equation.

5.5 Rényi and η-entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. It is defined by

$$I_\eta\left(X\right)=\frac{1}{1-\eta}\log\left(\left(J_\eta(X)\right),\ \eta>0 \text{ and } \eta\neq 1,\right.$$

where

$$J_\eta(X)=\int_{-\infty}^{\infty}f^\eta(x)dx.$$

Using (3.2), we can write

$$\begin{aligned} f^\eta(x) &= \frac{\left(2\theta\beta\alpha^\beta\right)^\eta x^{-\eta(\beta+1)}}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]\right\}^{3\eta}}\exp\left[-2\eta\left(\frac{\alpha}{x}\right)^\beta\right]\ \exp\left[-\eta\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right)^2\right] \\ &\times\left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right)^2\right]\right\}^{\eta(\theta-1)}. \end{aligned}$$

After some simplifications, we have

$$f^\eta(x)=\sum_{j,k=0}^{\infty}d_{j,k}\ \ x^{-\eta(\beta+1)}\exp\left\{-\left[2\left(\eta+j\right)+k\right]\left(\frac{\alpha}{x}\right)^\beta\right\},$$

where

$$d_{j,k}=\left(2\theta\beta\alpha^\beta\right)^\eta\sum_{i=0}^{\infty}\frac{(-1)^{i+j+k}}{j!\left(\eta+i\right)^{-j}}\binom{\eta\left(\theta-1\right)}{i}\binom{-3\eta-2j}{k}.$$

Now, $J_\eta(X)$ becomes

$$J_\eta\left(X\right)=\Gamma\left(\frac{\eta\left(\beta+1\right)-1}{\beta}\right)\sum_{j,k=0}^{\infty}p_{j,k}\left[2\left(\eta+j\right)+k\right]^{\frac{1-\eta(\beta+1)}{\beta}}\quad, \tag{5.5}$$

where

$$p_{j,k}=\left(2\theta\right)^\eta\left(\frac{\beta}{\alpha}\right)^{\eta-1}\sum_{i=0}^{\infty}\frac{(-1)^{i+j+k}}{j!\left(\eta+i\right)^{-j}}\binom{\eta\left(\theta-1\right)}{i}\binom{-3\eta-2j}{k}.$$

Hence, the Rényi entropy of $X \sim$BrXFr(α, β, θ) is

$$I_\eta(X) = \frac{1}{1-\eta} \log \left(\sum_{j,k=0}^{\infty} d_{j,k} \int_0^\infty x^{-\eta(\beta+1)} \exp\left\{ -\left[2(\eta+j)+k\right] \left(\frac{\alpha}{x}\right)^\beta \right\} dx \right).$$

The η-entropy, say $H_\eta(X)$, is defined by

$$H_\eta(X) = \frac{1}{\eta-1} \log\left[1 - J_\eta(X)\right], \ \eta > 0 \text{ and } \eta \neq 1,$$

and then it follows from (5.5).

5.6 Order statistics

Let $X_1, \ldots, X_n$ be a random sample of size n from the BrXFr distribution and $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. Then, the pdf of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \tag{5.6}$$

Using (3.1) and (3.2), we can write

$$\begin{aligned} f(x)F(x)^{i+j-1} &= \frac{2\theta\beta\alpha^\beta x^{-\beta-1} \exp\left[-2\left(\frac{\alpha}{x}\right)^\beta\right]}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]\right\}^3} \exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right)^2\right] \\ &\quad \times \left\{1-\exp\left[-\left(\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right)^2\right]\right\}^{\theta(i+j)-1}. \end{aligned}$$

Using (4.1), the exponential series and (4.3), and after some algebra, we have

$$\begin{aligned} f(x)F(x)^{i+j-1} &= 2\theta\beta\alpha^\beta x^{-\beta-1} \sum_{r,k,l=0}^{\infty} \frac{(-1)^{r+k+l}(r+1)^k}{k!} \binom{-2k-3}{l} \\ &\quad \times \binom{\theta(i+j)-1}{r} \exp\left[-\left[2(k+1)+l\right]\left(\frac{\alpha}{x}\right)^\beta\right]. \end{aligned} \tag{5.7}$$

By inserting (5.7) in (5.6), we can write

$$f_{i:n}(x) = \sum_{k,l=0}^{\infty} m_{k,l} \ g_{2(k+1)+l}(x), \tag{5.8}$$

where

$$m_{k,l} \quad = \quad \sum_{r=0}^{\infty}\sum_{j=0}^{n-i} \frac{(-1)^{j+r+k+l} 2\theta\,(r+1)^k}{k!\mathrm{B}(i,n-i+1)\,[2\,(k+1)+l]} \binom{n-i}{j}\binom{-2k-3}{l}\binom{\theta\,(i+j)-1}{r}.$$

and $g_{2(k+1)+l}(x)$ denotes the Fr density function with shape parameter β and scale parameter $\alpha[2\,(k+1)+l]^{1/\beta}$. Thus, the density function of the BrXFr order statistics is a linear combination of Fr densities. Based on equation (5.8), we can derive some properties of $X_{i:n}$ from those Fr properties.

The qth moment of $X_{i:n}$ (for $q<\alpha$) is given by

$$E(X_{i:n}^q) = \alpha^q \Gamma\left(1-\frac{q}{\beta}\right)\sum_{k,l=0}^{\infty} m_{k,l}\ \ [2\,(k+1)+l]^{q/\beta}.$$

6 Maximum likelihood estimation

The maximum likelihood is the most commonly employed method, in the literature, for parameter estimation because the maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. Further, the normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. The MLEs of the parameters of the BrXFr model is now discussed. Let $x_1,\ldots,x_n$ be a random sample of this distribution with unknown parameter vector $\varphi=(\alpha,\beta,\theta)^T$.

The log-likelihood function for φ, say $\ell=\ell(\varphi)$, is given by

$$\begin{aligned}\ell \quad = \quad & n\log(2)+n\log(\beta)+n\log(\theta)+n\beta\log(\alpha)-(\beta+1)\sum_{i=1}^{n}\log x_i \\ & -2\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}-3\sum_{i=1}^{n}\log\left[1-K(x_i)\right]-\sum_{i=1}^{n}\left[\frac{K(x_i)}{1-K(x_i)}\right]^2 \\ & +(\theta-1)\sum_{i=1}^{n}\log\left\{1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right\}, \qquad (6.1)\end{aligned}$$

where $K(x_i)=\exp\left[-(\alpha/x_i)^{\beta}\right].$

Equation (6.1) can be maximized either directly by using the `R` (`optim` function), `SAS` (`PROC NLMIXED`), `Ox` program (sub-routine `MaxBFGS`) or by solving the nonlinear likelihood equations obtained by differentiating (6.1).

The score vector elements, $\mathbf{U}(\varphi)=\frac{\partial\ell}{\partial\varphi}=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta}\right)^T$, are, respectively, given by

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{-2\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta} - \frac{3\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\frac{K(x_i)}{1-K(x_i)} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}}{\left[1-K(x_i)\right]^3} \\
& \frac{\beta n}{\alpha} - \frac{2\beta}{\alpha}(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}K^2(x_i)\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]}{\left[1-K(x_i)\right]^3\left\{1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right\}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + n\log(\alpha) - \sum_{i=1}^{n}\log(x_i) - 3\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\frac{K(x_i)\log\left(\frac{\alpha}{x_i}\right)}{1-K(x_i)} \\
& -2\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\log\left(\frac{\alpha}{x_i}\right) + 2\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\frac{K^2(x_i)\log\left(\frac{\alpha}{x_i}\right)}{\left[1-K(x_i)\right]^3} \\
& -2(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}K^2(x_i)\exp\left[-\left(\frac{K(x_i)}{1-K(x_i))}\right)^2\right]\log\left(\frac{\alpha}{x_i}\right)}{\left[1-K(x_i)\right]^3\left\{1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right\}},
\end{aligned}
$$

and

$$
\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right). \tag{6.2}
$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\varphi}) = 0$. By solving these equations simultaneously gives the MLEs $\widehat{\alpha}$ and $\widehat{\beta}$. They can not be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

For more simplicity, from (6.2) and for fixed α and β, we can obtain $\widehat{\theta}(\alpha,\beta)$ as

$$
\widehat{\theta}(\alpha,\beta) = -\frac{n}{\sum_{i=1}^{n}\log\left(1-\exp\left[-\left(\frac{K(x_i,\alpha,\beta)}{1-K(x_i,\alpha,\beta)}\right)^2\right]\right)}. \tag{6.3}
$$

The MLE of α and β denoted by $\widehat{\alpha}$ and $\widehat{\beta}$, respectively, these estimates can be obtained by numerically solving the following non-linear equations

$$n-2\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}-3\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\frac{K(x_i)}{1-K(x_i)}+2\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}}{\left(1-K(x_i)\right)^3}$$
$$-2(\widehat{\theta}(\alpha,\beta)-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}K^2(x_i)\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]}{\left[1-K(x_i)\right]^3\left\{1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right\}}=0$$

and

$$\frac{n}{\beta}+n\log(\alpha)-\sum_{i=1}^{n}\log\left(x_i\right)-3\sum_{i=1}^{n}\frac{K(x_i)\left(\frac{\alpha}{x_i}\right)^{\beta}\log\left(\frac{\alpha}{x_i}\right)}{1-K(x_i)}$$
$$-2\sum_{i=1}^{n}\left[\left(\frac{\alpha}{x_i}\right)^{\beta}\log\left(\frac{\alpha}{x_i}\right)\right]+2\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}K^2(x_i)\log\left(\frac{\alpha}{x_i}\right)}{\left[1-K(x_i)\right]^3}$$
$$-2(\widehat{\theta}(\alpha,\beta)-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}K^2(x_i)\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\log\left(\frac{\alpha}{x_i}\right)}{\left[1-K(x_i)\right]^3\left\{1-\exp\left[-\left(\frac{K(x_i)}{1-K(x_i)}\right)^2\right]\right\}}=0. \tag{6.4}$$

After the numerically iterative techniques are used to compute $\widehat{\alpha}$ and $\widehat{\beta}$ from (6.4), the MLE of α ($\widehat{\theta}(\alpha,\beta)$) can be computed from (6.3) as $\widehat{\theta}(\widehat{\alpha},\widehat{\beta})$. For the BrXFr distribution all the second order derivatives exist.

For interval estimation of the model parameters, we require the 3×3 observed information matrix $J(\varphi)=\{J_{rs}\}$ for $r,s=\alpha,\beta,\theta$. Under standard regularity conditions, the multivariate normal $N_3(0,J(\widehat{\varphi})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\varphi})$ is the total observed information matrix evaluated at $\widehat{\varphi}$. Therefore, approximate $100(1-\phi)\%$ confidence intervals for α,β and θ can be determined as:

$\widehat{\alpha}\pm z_{\frac{\phi}{2}}\sqrt{\widehat{J}_{\alpha\alpha}}$, $\widehat{\beta}\pm z_{\frac{\phi}{2}}\sqrt{\widehat{J}_{\beta\beta}}$ and $\widehat{\theta}\pm z_{\frac{\phi}{2}}\sqrt{\widehat{J}_{\theta\theta}}$, where $z_{\frac{\phi}{2}}$ is the upper ϕth percentile of the standard normal distribution.

7 Simulation study

We now present the results for some simulations that investigate the behavior of the MLEs in terms of the sample size n. All of these simulations were performed using R 3.3.3 programming language R (R Core Team (2017)). We generated $2,000$ random samples from the distribution BrXFr by using the relation (5.2) with three different sample sizes $n = 50$, $n = 150$ and $n = 400$. We set the true values of the parameters as follows: $\alpha = (0.5, 1, 2), \beta = (0.1, 1, 3)$ and $\theta = (0.4, 1, 5)$. For each sample size and each parameter combination, the average MLEs and mean square errors (MSEs) are computed. Tables 3, 4 and 5 show these results, it can be seen that the estimates are stable and quite close the true parameter values for these sample sizes. Furthermore, as the sample size increases the MSEs decreases in all cases.

8 Data analysis

This section is devoted to illustrate the empirical importance of the BrXFr distribution using an application to real data. The data set refers to the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia (Feigl and Zelen, 1965). The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

For this data set, we shall compare the fit of the BrXFr distribution with the MFr, WFr, KMOFr, KFr, EExFr, EFr, TEFr, GEFr, BFr, MOFr, TEGFr, BExFr, TFr and Fr distributions.

In order to compare the fitted distributions, we consider the following criteria: the $-2\widehat{\ell}$ (Maximized Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion). We also shall consider the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics. The model with minimum values for these statistics could be chosen as the best model to fit the data.

Table 6 provides the values of the MLEs and their corresponding standard errors (in parentheses) of the model parameters, whereas the values of these statistics for the fitted models to both data sets are listed in Table 7.

The plots of the fitted BrXFr, MFr, WFr, KMOFr, KFr and EExFr pdfs are displayed in Figure 4. The estimated cdfs for first six competitive models are shown in Figure 5. Figure 6 shows the QQ plots for these fitted models.

In order to assess if the model is appropriate, the plots of the fitted BrXFr density, estimated cdf and sf and the QQ plot for the BrXFr distribution are displayed in Figure 7. We conclude that the BrXFr distribution provides good fits to these data.

Table 3: Average values of MLEs and the corresponding MSEs ($n = 50$)

Parameters			MLE			MSE		
α	β	θ	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$
0.50000	0.10000	0.40000	0.56496	0.14271	0.23729	0.03294	0.00435	0.07150
		1.00000	0.51141	0.10252	1.01843	0.00216	0.00010	0.02564
		5.00000	0.58350	0.10189	5.09682	0.07572	0.00008	0.54922
	1.00000	0.40000	0.52458	1.04773	0.38918	0.02103	0.02863	0.01057
		1.00000	0.50351	1.02435	1.01390	0.00226	0.00850	0.02467
		5.00000	0.50470	1.02443	5.11664	0.00055	0.00968	0.59883
	3.00000	0.40000	0.92249	3.09378	0.23293	0.42310	0.04337	0.07065
		1.00000	0.70530	3.08053	0.83151	0.20462	0.05605	0.22435
		5.00000	0.50164	3.07991	5.11081	0.00006	0.08727	0.55255
1.00000	0.10000	0.40000	1.14806	0.14196	0.23854	0.16398	0.00427	0.07035
		1.00000	1.02135	0.10209	1.01788	0.00868	0.00009	0.02321
		5.00000	1.17407	0.10193	5.10225	0.36255	0.00009	0.54760
	1.00000	0.40000	1.01389	1.06128	0.39004	0.01612	0.03961	0.01145
		1.00000	1.00180	1.02150	1.01798	0.00008	0.00689	0.02186
		5.00000	1.00904	1.02323	5.11721	0.00218	0.00940	0.56880
	3.00000	0.40000	1.40192	3.08099	0.23833	0.40325	0.05051	0.06846
		1.00000	1.11342	3.07421	0.92060	0.11234	0.05984	0.13425
		5.00000	1.00255	3.06615	5.11568	0.00023	0.08380	0.58578
2.00000	0.10000	0.40000	2.24647	0.14419	0.23418	0.55362	0.00448	0.07356
		1.00000	2.03383	0.10243	1.01511	0.03554	0.00011	0.02645
		5.00000	2.41460	0.10247	5.10662	1.52793	0.00010	0.57877
	1.00000	0.40000	1.95415	1.18011	0.33845	0.03560	0.15802	0.03290
		1.00000	2.00327	1.02190	1.01809	0.00032	0.00688	0.02319
		5.00000	2.01914	1.02481	5.09743	0.00911	0.01003	0.58749
	3.00000	0.40000	2.06174	3.07204	0.38088	0.06440	0.05215	0.01472
		1.00000	2.00123	3.06149	1.01902	0.00004	0.06283	0.02146
		5.00000	2.00538	3.06889	5.10399	0.00094	0.08337	0.59510

Table 4: Average values of MLEs and the corresponding MSEs ($n = 150$)

Parameters			MLE			MSE		
α	β	θ	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$
0.50000	0.10000	0.40000	0.54684	0.15191	0.18915	0.00980	0.00533	0.08551
		1.00000	0.50382	0.10112	0.99871	0.00055	0.00007	0.01254
		5.00000	0.53336	0.10086	5.03348	0.02101	0.00003	0.17990
	1.00000	0.40000	0.54634	1.03832	0.37636	0.04103	0.02594	0.01155
		1.00000	0.50259	1.00847	1.00215	0.00181	0.00323	0.00987
		5.00000	0.50123	1.00653	5.03030	0.00017	0.00280	0.17776
	3.00000	0.40000	0.97562	3.03156	0.20775	0.47317	0.01876	0.07797
		1.00000	0.73382	3.02822	0.78327	0.23280	0.01782	0.24030
		5.00000	0.50045	3.02255	5.03531	0.00002	0.02471	0.16450
1.00000	0.10000	0.40000	1.09918	0.15134	0.19132	0.04178	0.00527	0.08471
		1.00000	1.00849	0.10124	0.99893	0.00212	0.00008	0.01287
		5.00000	1.07105	0.10093	5.02596	0.08519	0.00003	0.16702
	1.00000	0.40000	1.01478	1.06303	0.37707	0.01829	0.05233	0.01205
		1.00000	1.00084	1.00585	1.00292	0.00030	0.00233	0.00726
		5.00000	1.00306	1.00757	5.03991	0.00069	0.00278	0.17118
	3.00000	0.40000	1.43977	3.01523	0.22064	0.43589	0.02627	0.07308
		1.00000	1.18153	3.01855	0.83548	0.18003	0.01991	0.18851
		5.00000	1.00070	3.01948	5.04793	0.00008	0.02634	0.16796
2.00000	0.10000	0.40000	2.17564	0.15315	0.18453	0.14138	0.00545	0.08731
		1.00000	2.01612	0.10093	0.99936	0.00859	0.00007	0.01155
		5.00000	2.12501	0.10078	5.03305	0.32844	0.00003	0.18197
	1.00000	0.40000	1.90074	1.34178	0.24959	0.06775	0.31953	0.06232
		1.00000	2.00084	1.00637	1.00542	0.00009	0.00213	0.00738
		5.00000	2.00488	1.00641	5.02619	0.00258	0.00258	0.16889
	3.00000	0.40000	2.18326	3.01485	0.32559	0.17952	0.02950	0.03227
		1.00000	2.00034	3.02715	1.00484	0.00001	0.02071	0.00690
		5.00000	2.00244	3.02809	5.02601	0.00032	0.02623	0.17637

Table 5: Average values of MLEs and the corresponding MSEs ($n = 400$)

Parameters			MLE			MSE		
α	β	θ	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$
0.50000	0.10000	0.40000	0.52728	0.15183	0.18903	0.00369	0.00526	0.08476
		1.00000	0.50165	0.10181	0.98695	0.00066	0.00018	0.01851
		5.00000	0.51200	0.10032	5.01750	0.00662	0.00001	0.06462
	1.00000	0.40000	0.53170	1.03427	0.37864	0.02864	0.02717	0.00928
		1.00000	0.50532	1.00781	0.99588	0.00389	0.00489	0.01070
		5.00000	0.50074	1.00339	5.00699	0.00006	0.00095	0.06212
	3.00000	0.40000	0.95042	3.00668	0.21776	0.44781	0.01186	0.07327
		1.00000	0.74325	3.00711	0.76762	0.24224	0.00774	0.24668
		5.00000	0.50027	3.01191	5.01016	0.00001	0.00877	0.06458
1.00000	0.10000	0.40000	1.05645	0.14920	0.19992	0.01229	0.00500	0.08037
		1.00000	1.00293	0.10197	0.98729	0.00071	0.00019	0.02055
		5.00000	1.02970	0.10042	5.01293	0.02761	0.00001	0.06352
	1.00000	0.40000	1.01156	1.06977	0.36810	0.02264	0.06053	0.01356
		1.00000	1.00088	1.00454	1.00173	0.00027	0.00253	0.00464
		5.00000	1.00070	1.00196	5.00606	0.00026	0.00102	0.06063
	3.00000	0.40000	1.43997	2.98710	0.21931	0.43366	0.02354	0.07286
		1.00000	1.20589	3.00096	0.80487	0.20411	0.00945	0.21010
		5.00000	1.00023	3.00618	5.01093	0.00003	0.00908	0.06381
2.00000	0.10000	0.40000	2.10608	0.15188	0.18977	0.04473	0.00526	0.08452
		1.00000	2.00487	0.10163	0.98827	0.00284	0.00015	0.01702
		5.00000	2.04042	0.10023	5.01389	0.11010	0.00001	0.05986
	1.00000	0.40000	1.86057	1.42197	0.21021	0.08298	0.39228	0.07638
		1.00000	2.00030	1.00460	1.00112	0.00017	0.00263	0.00471
		5.00000	2.00307	1.00374	5.01160	0.00100	0.00100	0.06254
	3.00000	0.40000	2.30803	2.97575	0.27203	0.30147	0.02657	0.05221
		1.00000	2.00010	3.00758	1.00431	0.00000	0.00686	0.00269
		5.00000	2.00081	3.00966	5.01990	0.00011	0.00913	0.06634

Table 6: MLEs and their standard errors (in parentheses) for Leukaemia data

Model	Estimates				
	$\widehat{\alpha}$	$\widehat{\beta}$			
Fr	7.8652	0.6944			
	(2.0913)	(0.0915)			
	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$		
BrXFr	62.2609	0.5805	0.1715		
	(3.2345)	(0.0853)	(0.0512)		
MFr	14.2312	0.4634	0.0123		
	(7.4633)	(0.1139)	(0.0051)		
EFr	1426.6289	0.24909	13.7467		
	(3607.173)	(0.0708)	(13.5121)		
MOFr	1.9073	0.9875	8.0679		
	(1.8016)	(0.1859)	(11.1477)		
TFr	5.5489	0.7401	-0.4291		
	(2.9837)	(0.0995)	(0.5549)		
	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	
WFr	2.62229	1.8389	0.1704	0.3807	
	(1.8352)	(1.4657)	(0.1459)	(0.2714)	
KFr	9378.570	0.0842	5.5132	7160.57	
	(804.3827)	(0.0247)	(2.2439)	(17494.23)	
EExFr	401.2899	0.5184	0.1882	6.0288	
	(2.0789)	(0.1424)	(0.1587)	(2.6705)	
TEFr	1250.6801	0.2499	13.4874	-0.1175	
	(3390.324)	(0.0761)	(13.8716)	(0.4874)	
BFr	24.2231	0.0884	33.5337	60.5680	
	(305.1054)	(0.119015)	(111.4965)	(161.2934)	
GEFr	62.7173	0.0555	189.5528	79.3368	
	(1756.0625)	(0.1070)	(738.3019)	(388.7273)	
	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\delta}$	$\widehat{a}$	$\widehat{b}$
KMOFr	31946.73	0.6074	4.8067	1.0146	13724.51
	(579.3846)	(0.1067)	(6.1015)	(0.1390)	(10687.72)
TEGFr	309.5304	0.1925	-0.1872	10.3543	3.7951
	(518.1863)	(0.1109)	(0.5069)	(5.1201)	(8.0303)
BExFr	0.1162	4.3641	0.0437	9.3849	6.4056
	(0.0265)	(0.0253)	(0.0489)	(2.7092)	(9.6967)

Table 7: Goodness-of-fit statistics for Leukaemia data

Model	$-2\widehat{\ell}$	AIC	$CAIC$	BIC	$HQIC$	W^*	A^*
BrXFr	302.157	308.157	308.985	312.647	309.668	0.06539	0.42675
MFr	303.124	309.124	309.951	313.613	310.634	0.07108	0.47049
WFr	302.577	310.577	312.006	316.564	312.592	0.06134	0.42550
KMOFr	304.804	314.804	317.026	322.286	317.321	0.08367	0.54694
KFr	304.832	314.832	316.261	320.818	316.846	0.09461	0.63420
EExFr	306.680	314.680	316.109	320.666	316.694	0.10293	0.64751
EFr	307.788	313.788	314.616	318.277	315.299	0.11151	0.70509
TEFr	307.760	315.760	317.189	321.746	317.774	0.11041	0.70069
GEFr	307.861	315.861	317.289	321.847	317.875	0.11385	0.71476
BFr	307.991	315.991	317.420	321.978	318.006	0.11569	0.72387
MOFr	309.378	315.378	316.206	319.868	316.889	0.12888	0.79777
TEGFr	308.893	318.893	321.115	326.376	321.411	0.12552	0.77657
BExFr	309.905	319.905	322.127	327.387	322.422	0.13931	0.85497
TFr	311.449	317.449	318.276	321.938	318.959	0.15502	0.94183
Fr	311.997	315.997	316.397	318.990	317.004	0.16011	0.97592

In Table 7, we compare the BrXFr model with the MFr, WFr, KMOFr, KFr, EExFr, EFr, TEFr, GEFr, BFr, MOFr, TEGFr, BExFr, TFr and Fr distributions. Its noted that the proposed model has the lowest values for the AIC, $CAIC$, $HQIC$, BIC, W^* and A^* statistics among all fitted models (except W^* and A^* for the WFr model). So, the BeXFr model can be chosen as the best model among all fitted models.

9 Conclusions

In this paper, we propose a three-parameter model, called the Burr X Fréchet (BrXFr) distribution, which extends the Fréchet (Fr) distribution pioneered by Maurice Fréchet (1878-1973). An obvious reason for generalizing a standard distribution is the fact that the generated model can provide more flexibility to analyze real life data. We provide some of its mathematical and statistical properties. The BrXFr density function can be expressed as a mixture of Fr densities. We derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions and moments of the residual life and reversed residual life model, Rényi and η-entropies and order statistics. We discuss the estimation of the model parameters by maximum likelihood. The proposed distribution is applied to a

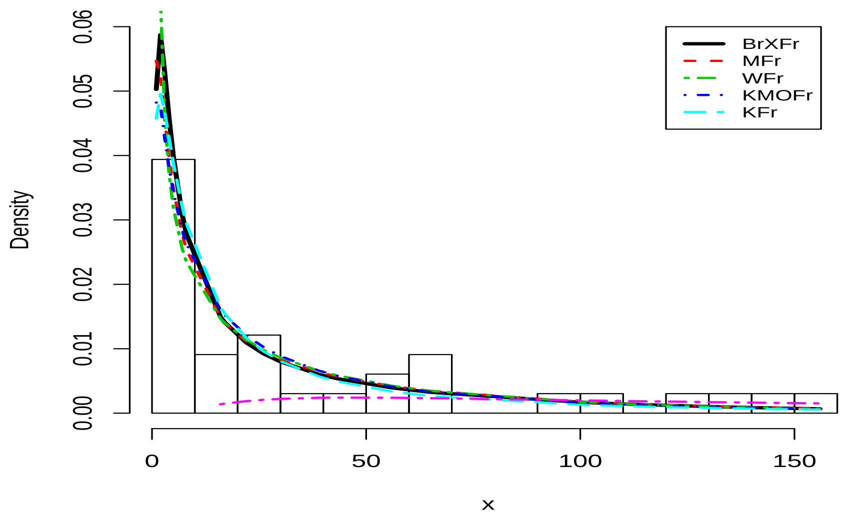


Figure 4: The fitted pdfs of the BrXFr, MFr, WFr, KMOFr, KFr and EExFr models for Leukaemia data.

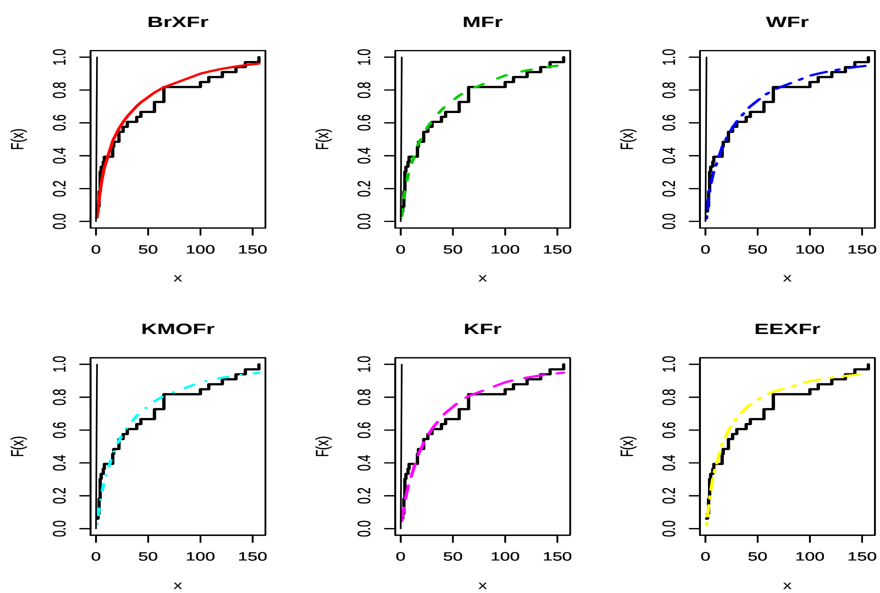


Figure 5: Fitted cdfs of the BrXFr, MFr, WFr, KMOFr, KFr and EExFr models for Leukaemia data.

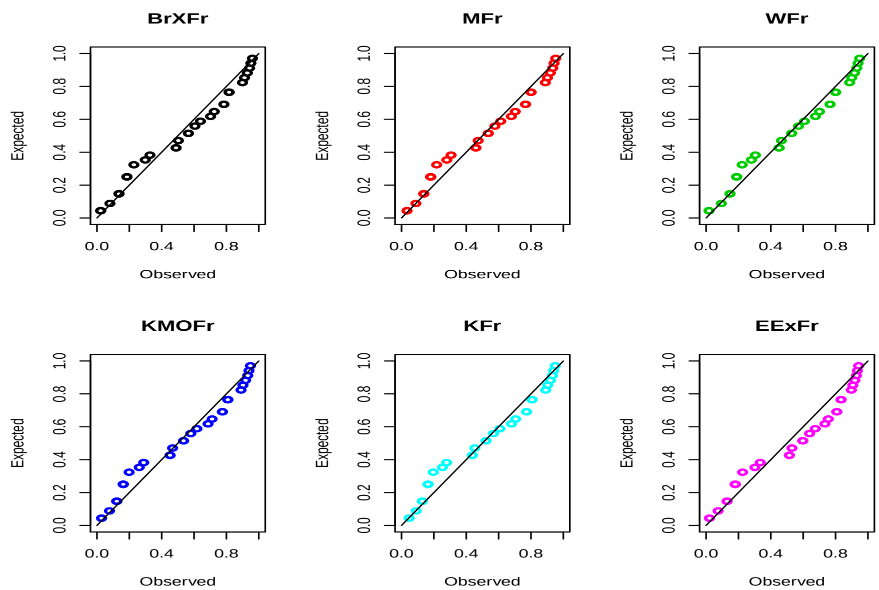


Figure 6: Q-Q plots of the BrXFr, MFr, WFr, KMOFr, KFr and EExFr models for Leukaemia data.

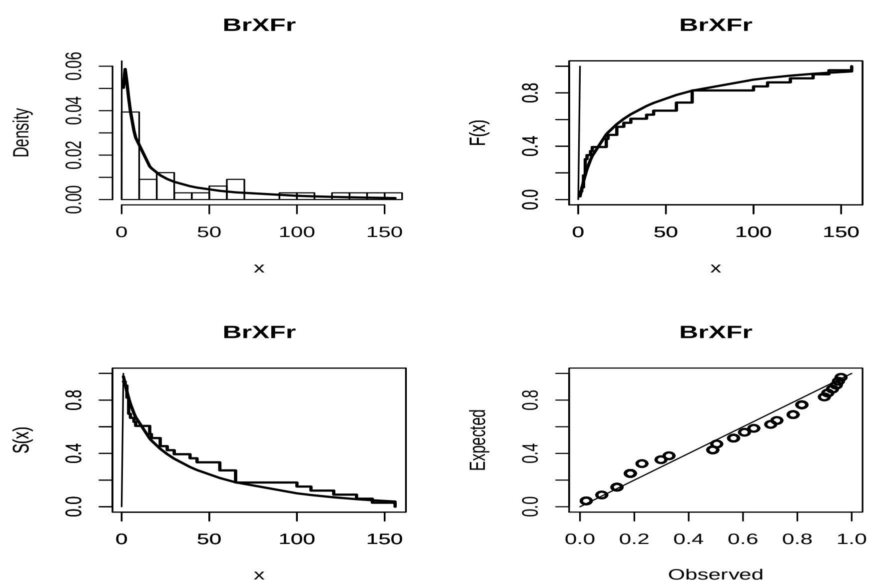


Figure 7: The fitted pdf, estimated cdf and sf and the Q-Q plot of the BrXFr distribution.

real data set. It provides a better fit than several other competitive nested and non-nested models. We hope that the proposed model will attract wider application in areas such as survival and lifetime data, engineering, hydrology, economics.

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