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THE KUMARASWAMY MARSHALL-OLKIN FRÉCHET DISTRIBUTION WITH APPLICATIONS

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ABSTRACT

The Fréchet distribution is an absolutely continuous distribution with wide applicability in extreme value theory. Generalizing distributions is always precious for applied statisticians and recent literature has suggested several ways of extending well-known distributions. We propose a new lifetime model called the Kumaraswamy Marshall–Olkin Fréchet distribution, which generalizes the Marshall–Olkin Fréchet distribution and at least seventeen known and unknown lifetime models. Various properties of the new model are explored including closed-forms expressions for moments, quantiles, generating function, order statisics and Rényi entropy. The maximum likelihood method is used to estimate the model parameters. We compare the flexibility of the proposed model with other related distributions by means of two real data sets.

KEYWORDS

Generating Function, Marshall–Olkin Fréchet, Maximum Likelihood, Moment, Order Statistics, Rényi Entropy.

1. INTRODUCTION

The procedure of expanding a family of distributions to add flexibility or to construct covariate models is a well-known technique in the literature. In many applied sciences like medicine, engineering and finance, among others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model these types of data.

Marshall and Olkin (2007) defined and studied a new method of adding a new parameter to an existing distribution. The generated distribution includes the baseline distribution as a special case and gives more flexibility to model real data. For further and comprehensive information about the Marshall–Olkin family of distributions, see Barreto-Souza et al. (2013).

The Fréchet (Fr) distribution is an important distribution developed within the general extreme value theory, which deals with the stochastic behaviour of the maximum and the minimum of independent and identically distributed random variables. It has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea waves and wind speeds. It has been applied to data on characteristics of sea waves, wind speeds, etc. For more information about the Fr distribution and its applications, see Kotz and Nadarajah (2000).

Given its applicability in extreme value theory, several extensions of the Fr distribution were proposed in the literature. Some of its applications in various fields are given in Harlow (2002). He showed that it is an important distribution for modeling statistical behavior of material properties for a variety of engineering applications. Nadarajah and Kotz (2008) discussed the sociological models based on Fr random variables. Zaharim et al. (2009) applied the Fr distribution for analyzing the wind speed data. Mubarak (2011) studied the Fr progressive type-II censored data with binomial removals.

Krishna et al. (2013) proposed some applications of the Marshall–Olkin Fr (MOFr) distribution such as the reliability test plan for accepting or rejecting a lot where the lifetime of the product follows the MOFr distribution and showed that it has also an application in time series modeling.

Many authors defined generalizations of the Fr distribution. For example, Nadarajah and Kotz (2003) pioneered the exponentiated Fr, Nadarajah and Gupta (2004) studied the beta Fr, Mahmoud and Mandouh (2013) defined and studied the transmuted Fr, Krishna et al. (2013) proposed the MOFr, Mead and Abd-Eltawab (2014) investigated the Kumaraswamy Fr, Afify et al. (2015) defined the transmuted Marshall–Olkin Fr, Mead et al. (2016) introduced the beta exponential Fr and Afify et al. (2016) proposed the Weibull Fr distributions.

The cumulative distribution function (cdf) of the MOFr distribution (for $x > 0$) is given by

$$G(x) = \exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-1}, \tag{1}$$

where $\delta > 0$ is a scale parameter and $\alpha > 0$ and $\beta > 0$ are shape parameters. The corresponding probability density function (pdf) is given by

$$g(x) = \alpha\beta\delta^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-2}. \tag{2}$$

Our aim is to define and study a new lifetime model called the *Kumaraswamy Marshall-Olkin Fréchet* (Kw-MOFr) distribution. The main feature of this model is that two additional shape parameters are inserted in (1) to give more flexibility in the form of the generated distribution. Following the Kumaraswamy-generalized (Kw-G) class pioneered by Cordeiro and de Castro (2011), we construct the five-parameter Kw-MOFr model. We give a comprehensive description of some mathematical properties of the new distribution with the hope that it will attract wider applications in reliability, engineering and other research.

For an arbitrary baseline cdf, $G(x)$, Cordeiro and de Castro (2011) defined the Kw-G family of distributions by the cdf and pdf given by

$$F(x) = 1 - [1 - G(x)^a]^b \tag{3}$$

and

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1}, \tag{4}$$

respectively, where $g(x) = dG(x)/dx$ and a and b are two additional positive shape parameters. They can provide greater control over the weights in both tails and in the center of the generated distribution and govern skewness. Clearly, for $a = b = 1$, we obtain the baseline distribution. If X is a random variable with pdf (4), we write $X \sim$Kw-G(a, b).

To this end, we start from the MOFr model to define the new Kw-MOFr model by inserting (1) and (2) in equations (3) and (4). Then, we obtain the cdf and pdf of the Kw-MOFr distribution.

The rest of the paper is organized as follows. In Section 2, we define the Kw-MOFr distribution, provide some plots of its pdf and hazard rate function (hrf) and list some special cases. We determine linear representations for the pdf and cdf in Section 3. Some mathematical properties of the Kw-MOFr model including quantiles, ordinay moments and cumulants, moment generating function (mgf), incomplete moments, mean deviations, Rényi and q-entropies, moments of the residual life and reversed residual life are obtained in Section 4. In Section 5, the order statistics and their moments are determined. The maximum likelihood estimates (MLEs) of the unknown parameters are explored in Section 6. We fit the new distribution to two real data sets to examine the performance of the new model using nested and non-nested distributions in Section 7. Finally, in Section 8, we provide some conclusions.

2. THE Kw-MOFr MODEL

The cdf of the Kw-MOFr distribution (for $x > 0$) is given by

$$F(x) = 1 - \left\{1 - \exp\left[-\alpha\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a}\right\}^{b}, \tag{5}$$

and its pdf can be reduced to

$$\begin{aligned} f(x) = {} & \alpha\beta ab\delta^{\beta}x^{-\beta-1}\exp\left[-\alpha\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a-1} \\ & \times\left\{1 - \exp\left[-\alpha\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a}\right\}^{b-1}. \end{aligned} \tag{6}$$

Henceforth, we denote a random variable X having the pdf (6) by $X \sim$Kw-MOFr$(\alpha, \beta, \delta, a, b)$.

The survival function (sf), hrf and cumulative hazard rate function (chrf) of X are, respectively, given by

$$S(x) = \left\{1 - \exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a}\right\}^{b},$$

$$h(x) = \frac{\alpha\beta ab\delta^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha+(1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a-1}}{1-\exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha+(1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a}}$$

and

$$H(x) = -b\log\left\{1 - \exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a}\right\}.$$

Plots of the Kw-MOFr pdf for some parameter values are displayed in Figure 1. Figure 2 displays some possible shapes of the hrf of the Kw-MOFr model for selected parameter values.

The Kw-MOFr model is a very flexible distribution that approaches different distributions when its parameters are changed. The new distribution includes as special cases seventeen well-known and other new models, namely the Kumaraswamy Fr (Kw-Fr), Kumaraswamy inverse exponential (Kw-IE), Kumaraswamy inverse Rayleigh (Kw-IR), Kumaraswamy Marshall–Olkin inverse exponential (Kw-MOIE), Kumaraswamy Marshall–Olkin inverse Rayleigh (Kw-MOIR), MOFr, Marshall–Olkin inverse exponential (MOIE), Marshall–Olkin inverse Rayleigh (MOIR), exponentiated Fr (EFr), exponentiated inverse exponential (EIE), exponentiated inverse Rayleigh (EIR), generalized Fr (GFr), generalized inverse exponential (GIE), generalized inverse Rayleigh (GIR), Fr, inverse exponential (IE), inverse Rayleigh (IR) models. Table 1 lists all these sub-models.

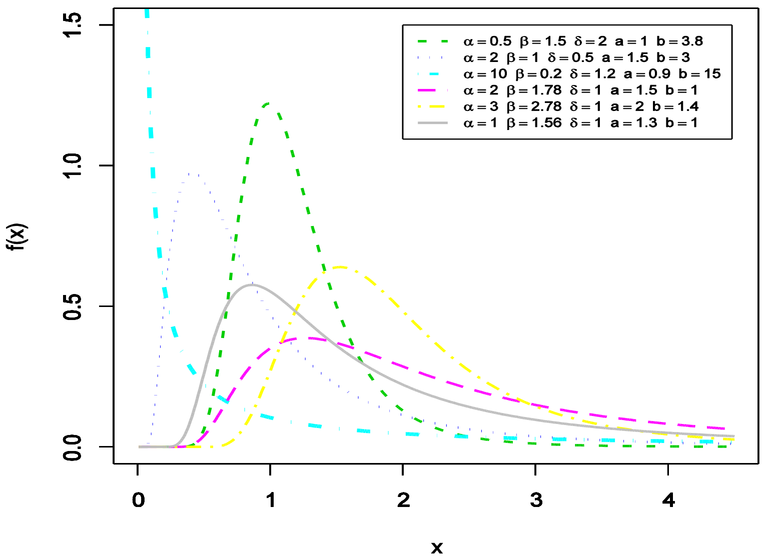


Figure 1: Plots of the Kw-MOFr Density Function

Table 1
Sub-Models of the Kw-MOFr Distribution

No.	Distribution	α	β	δ	a	b	Author
1	Kw-Fr	1	β	δ	a	b	Mead and Abd-Eltawab (2014)
2	Kw-IE	1	1	δ	a	b	–
3	Kw-IR	1	2	δ	a	b	–
4	Kw-MOIE	α	1	δ	a	b	New
5	Kw-MOIR	α	2	δ	a	b	New
6	MOFr	α	β	δ	1	1	Krishna et al. (2013)
7	MOIE	α	1	δ	1	1	–
8	MOIR	α	2	δ	1	1	–
9	EFr	1	β	δ	1	b	Nadarajah and Kotz (2003)
10	EIE	1	1	δ	1	b	–
11	EIR	1	2	δ	1	b	–
12	GFr	1	β	δ	a	1	New
13	GIE	1	1	δ	a	1	New
14	GIR	1	2	δ	a	1	New
15	Fr	1	β	δ	1	1	Fréchet (1924)
16	IE	1	1	δ	1	1	Keller and Kamath (1982)
17	IR	1	2	δ	1	1	Trayer (1964)

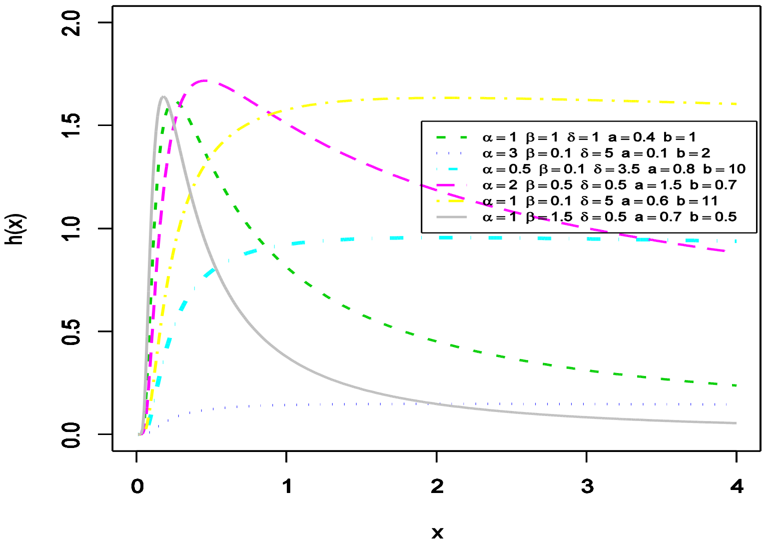


Figure 2: Plots of the Kw-MOFr hrf

3. LINEAR REPRESENTATION

In this section, we provide linear representations for the pdf and cdf of the new distribution. Consider a power series given by

$$(1-z)^{-\alpha}=\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)j!}\,z^{j},|z|<1,\alpha>0. \tag{7}$$

Applying expansion (7) to equation (6) gives

$$f(x)=\alpha\,\beta\,a\,b\,\delta^{\beta}\,x^{-\beta-1}\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(b)}{j!\Gamma(b-j)}\exp\left[-a(j+1)\left(\frac{\delta}{x}\right)^{\beta}\right]$$
$$\times\left\{\alpha+(1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right\}^{-a(j+1)-1}. \tag{8}$$

Applying (7) again to (8), we obtain

$$f(x)=\sum_{j,i=0}^{\infty}\frac{(-1)^{j}\Gamma(b)\Gamma(aj+a+i+1)}{j!i!\Gamma(b-j)\Gamma(aj+a+1)}\alpha^{-a(j+1)}\left(1-\frac{1}{\alpha}\right)^{i}\beta\,a\,b\,\delta^{\beta}$$
$$\times x^{-\beta-1}\exp\left[-(aj+a+i)\left(\frac{\delta}{x}\right)^{\beta}\right].$$

The last equation can be expressed as

$$f(x)=\sum_{j,i=0}^{\infty}\upsilon_{j,i}\,h_{aj+a+i}(x), \tag{9}$$

where $h_{aj+a+i}(x)$ is the Fr density with scale parameter $\delta(aj+a+i)^{1/\beta}$ and shape parameter β and $\upsilon_{j,i}$ is a constant term given by

$$\upsilon_{j,i}=\frac{(-1)^{j}a\Gamma(b+1)\Gamma(aj+a+i)}{\alpha^{a(j+1)}\,j!\,i!\,\Gamma(b-j)\Gamma(aj+a+1)}\left(1-\frac{1}{\alpha}\right)^{i}.$$

Therefore, the Kw-MOFr density is given by a linear combination of the Fr densities. So, several of its structural properties can be derived from those of the Fr distribution.

Similarly, the cdf of X in (5) admits a linear representation given by

$$F(x)=\sum_{j,i=0}^{\infty}\upsilon_{j,i}\,H_{aj+a+i}(x),$$

where $H_{aj+a+i}(x)$ is the cdf of the Fr model with scale parameter $\delta(aj+a+i)^{1/\beta}$ and shape parameter β.

These linear representations will be used to obtain some structural properties of the new distribution.

4. SOME STATISTICAL PROPERTIES

Established algebraic expansions to determine some structural properties of the Kw-MOFr distribution can be more efficient than computing those directly by numerical integration of its density function. The statistical properties of the Kw-MOFr distribution including moments, quantile function (qf), mgf, incomplete moments, mean deviations and Rényi entropy are discussed in the following sections.

4.1 Moments

Let Y be a random variable having the Fr distribution with parameters δ and β. The rth moment of Y (for $r < \beta$) is given by $\mu_r' = \delta^r \Gamma\left(1 - \frac{r}{\beta}\right)$ and the sth incomplete moment of Y is given by $\varphi_s(t) = \delta^s \gamma\left(1 - \frac{s}{\beta}, \left(\frac{\delta}{t}\right)^{\beta}\right)$, where $\Gamma(b) = \int_0^\infty y^{b-1}\, \mathrm{e}^{-\mathrm{y}} \mathrm{dy}$ is the complete gamma function and $\gamma(b,z) = \int_0^z y^{b-1}\, \mathrm{e}^{-\mathrm{y}} \mathrm{dy}$ is the lower incomplete gamma function.

The rth ordinary moment, say μ_r', of X is given by

$$\mu_r' = E(X^r) = \sum_{j,i=0}^{\infty} \upsilon_{j,i} \int_0^\infty x^r\, h_{aj+a+i}(x)dx.$$

Thus, we obtain (for $r < \beta$)

$$\mu_r' = E(X^r) = \delta^r \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{j,i=0}^{\infty} \upsilon_{j,i} (aj + a + i)^{r/\beta}. \tag{10}$$

By setting $r = 1$ in equation (10), we obtain the mean of X. The skewness and kurtosis measures can be determined from the ordinary moments of X using well-known relationships.

The nth central moment of X, say μ_n, is just given by

$$\mu_n = E(X - \mu_1')^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r}\, \mu_r'.$$

Then,

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r}\, \delta^r\, \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{j,i=0}^{\infty} \upsilon_{j,i} (aj + a + i)^{r/\beta}.$$

The cumulants (κ_s) of X follow from (10) as $\kappa_s = \mu_s' - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k\, \mu_{s-k}'$, respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3{\mu_2'}^2 + 12\mu_2'{\mu_1'}^2 - 6{\mu_1'}^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be calculated from the third and fourth standardized cumulants.

Finally, the pth descending factorial moment of X is $\mu_{(p)}' = (X^{(p)}) = [X(X-1) \times \cdots \times (X - p + 1)] = \sum_{n=0}^{p} s(p,n)\, \mu_n'$, where $s(r,n) = (n!)^{-1}[d^n n^{(r)}/dx^n]_{x=0}$ is the Stirling number of the first kind.

4.2 Quantile and Generating Functions

The qf of X is determined by inverting (5) as

$$Q(u) = \delta \left\{ \log\left(1 + \frac{1 - \sqrt[a]{1 - \sqrt[b]{1-u}}}{\alpha \sqrt[a]{1 - \sqrt[b]{1-u}}} \right) \right\}^{-1/\beta}, 0 < u < 1.$$

Simulating the Kw-MOFr random variable is straightforward. If U is a uniform variate in the unit interval $(0,1)$, the random variable $X = Q(U)$ follows (6).

The effects of the additional shape parameters α, a and b on the skewness and kurtosis of the new distribution can be based on quantile measures. The well-known Bowley's skewness and Moors' kurtosis are, respectively, defined by

$$\mathcal{B} = \frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$

and

$$\mathcal{M} = \frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments. Figure 3 displays some plots for the measures $\mathcal{B}$ and $\mathcal{M}$ of X. They indicate the variability of these measures on the shape parameters. It is clear from Figure 3 that the Kw-MOFr model is a right-skewed distribution.

Next, we provide the mgf of the Fr distribution as defined by Afify et al. (2016). We can write the mgf of Y as

$$M(t;\beta,\delta) = \beta\delta^{\beta}\int_0^{\infty} e^{t/y}\, y^{(\beta-1)}\, e^{-(\delta y)^{\beta}} dy.$$

By expanding the first exponential and determining the integral, we obtain

$$M(t;\beta,\delta) = \sum_{m=0}^{\infty} \frac{\delta^m t^m}{m!}\, \Gamma\left(\frac{\beta-m}{\beta}\right).$$

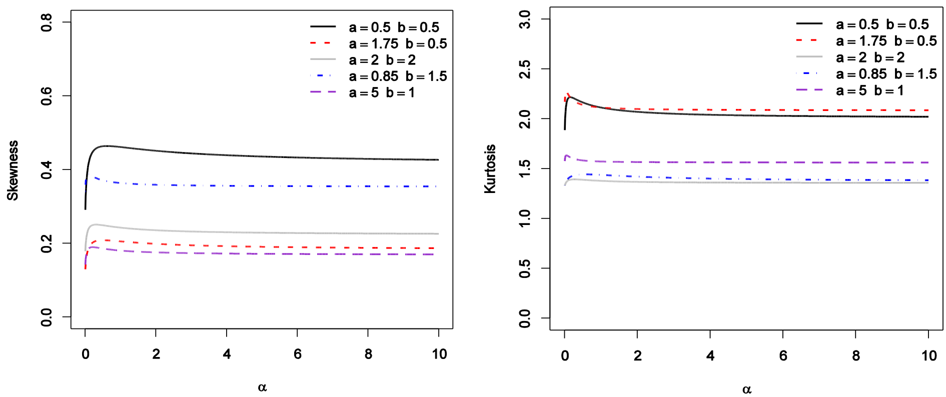


Figure 3: Skewness (left panel) and Kurtosis (right panel) Plots of the Kw-MOFr Distribution

Consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{matrix}(\gamma_1,A_1),\ldots,(\gamma_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix};\,x\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\gamma_j+A_j\,n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j\,n)}\,\frac{x^n}{n!}.$$

Then, we can write $M(t;\beta,\delta)$ as

$$M(t;\beta,\delta) ={}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\ -\end{matrix};\,\delta\, t\right].$$

Combining the last expression and (9), the mgf of X reduces to

$$M_X(t) = \sum_{j,i=0}^{\infty} \upsilon_{j,i}\ {}_1\Psi_0 \left[\begin{matrix} (1, -\beta^{-1}) \\ - \end{matrix} ;\ \delta(aj + a + i)^{1/\beta}\, t \right].$$

4.3 Incomplete Moments

The sth incomplete moment of X, denoted by $\varphi_s(t)$, is given by $\varphi_s(t) = \int_0^t x^s f(x)dx$. We can write from equation (9)

$$\varphi_s(t) = \sum_{j,i=0}^{\infty} \upsilon_{j,i} \int_0^t x^s\, h_{aj+a+i}(x),$$

and using the lower incomplete gamma function, we obtain (for $s < \beta$)

$$\varphi_s(t) = \delta^r \sum_{j,i=0}^{\infty} \upsilon_{j,i}(aj + a + i)^{r/\beta} \gamma\left(1 - \frac{s}{\beta}, (aj + a + i)\left(\frac{\delta}{t}\right)^{\beta}\right). \qquad (11)$$

The first incomplete moment of X is immediately obtained from (11) by setting $s = 1$. The main application of $\varphi_1(t)$ is related to the mean residual life and the mean waiting time (also known as mean inactivity time) given by $m_1(t;\theta) = [1 - \varphi_1(t)]/S(t) - t$ and $M_1(t;\theta) = t - \varphi_1(t)/F(t)$, respectively.

Further, the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median M $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = \int_0^{\infty} |X - \mu_1'|\, f(x)dx = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \int_0^{\infty} |X - M|\, f(x)dx = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$ comes from (10) and $F(\mu_1')$ is determined by (5).

Another application of $\varphi_1(t)$ refers to the Lorenz and Bonferroni curves defined by $L(p) = \varphi_1(q)/\mu_1'$ and $B(p) = \varphi_1(q)/(p\mu_1')$, respectively, where $q = Q(p)$ can be computed for a given probability p by inverting (5) numerically.

4.4 Rényi and δ-Entropies

Entropy refers to the amount of uncertainty associated with a random variable. The Rényi entropy has numerous applications in information theory (for example, classification, distribution identification problems, and statistical inference), computer science (for example, average case analysis for random databases, pattern recognition, and image matching) and econometrics, see Källberg et al. (2014). The Rényi entropy of a random variable X is defined by

$$I_\delta(X) = \frac{1}{1-\delta} \log\left(\int_{-\infty}^{\infty} f^{\delta}(x)dx\right), \delta > 0 \text{ and } \delta \neq 1.$$

Then, the Rényi entropy for the Kw-MOFr model reduces to

$$I_\delta(X) = \frac{1}{1-\delta} \log\left\{ T \sum_{i,j=0}^{\infty} \rho_{j,i} \underbrace{\int_0^{\infty} x^{-\delta(1+\beta)} \exp\left[-(a\delta + aj + i)\left(\frac{\delta}{x}\right)^{\beta}\right] dx}_{A} \right\},$$

where

$$T = \left(\beta ab\, \delta^{\beta}\right)^{\delta} \text{ and } \rho_{j,i} = \frac{(-1)^j (ab)^{\delta} \beta^{\delta-1} \delta^{1-\delta} \Gamma(\delta b - \delta + 1)\Gamma(aj + a\delta + \delta + i)}{j!\,i!\,\alpha^{a\delta+aj} \Gamma(\delta b - (\delta + j) + 1)\Gamma(aj + a\delta + \delta)} \left(1 - \frac{1}{\alpha}\right)^{i}.$$

But

$$A = \frac{1}{\beta}\delta^{1-\delta(1+\beta)}(a\delta + aj + i)^{[1-\delta(1+\beta)]/\beta}\Gamma(p),$$

where $p = (\delta\beta + \delta - 1)/\beta$.

Finally, we obtain

$$I_\delta(X) = \frac{1}{1-\delta}\log\{\sum_{i,j=0}^{\infty} \rho_{j,i}\, (a\delta + aj + i)^{-p}\Gamma(p)\}.$$

The δ-entropy, say $H_\delta(X)$, is defined by

$$H_\delta(X) = \frac{1}{\delta-1}\log\{1 - [(1-\delta)I_\delta(X)]\}.$$

4.5 Moments of the Residual and Reversed Residual Life

The nth moment of the residual life, $m_n(t) = E((X-t)^n|X > t)$, $n = 1,2, \ldots$, uniquely determines $F(x)$ (see Navarro et al., 1998). It is given by $m_n(t) = \frac{1}{1-F(t)}\int_t^\infty (x-t)^n dF(x)$.

Using equation (9), we can write

$$m_n(t) = \frac{1}{S(t)}\sum_{j,i=0}^{\infty} \upsilon_{j,i}\int_t^\infty (x-t)^n h_{aj+a+i}(x)dx.$$

By using the binomial expansion and the upper incomplete gamma function defined by $\Gamma(a,x) = \int_x^\infty y^{a-1}\,\mathrm{e}^{-y}\mathrm{d}y$, we obtain

$$m_n(t) = \frac{1}{S(t)}\sum_{r=0}^{n} \frac{(-1)^{n-r}\, n!\, t^{n-r}\delta^r}{r!\,(n-r)!\,(aj+a+i)^{-r/\beta}}\sum_{j,i=0}^{\infty} \rho_{j,i}\,\Gamma\left(1-\frac{r}{\beta}, (aj+a+i)\left(\frac{\delta}{t}\right)^\beta\right).$$

Another interesting function is the mean residual life (MRL) function at age t defined by $m_1(x) = E[(X-x)|X > x]$, which represents the expected additional life length for a unit which is alive at age x. The MRL of X follows by setting $n = 1$ in the last equation.

The nth moment of the reversed residual life, $M_n(t) = E[(t-X)^n|X \le t]$, for $t > 0, n = 1,2, \ldots$, uniquely determines $F(x)$. The nth moment of the residual life is given by $M_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n dF(x)$.

Using equation (9), we can write

$$m_n(t) = \frac{1}{F(t)}\sum_{j,i=0}^{\infty} \upsilon_{j,i}\int_0^t (t-x)^n h_{aj+a+i}(x)dx.$$

By using the binomial expansion and the lower incomplete gamma function, we obtain

$$M_n(t) = \frac{1}{F(t)}\sum_{r=0}^{n} \frac{(-1)^r\, n!\; t^{n-r}}{r!\;(n-r)!}\,\delta^r(aj+a+i)^{r/\beta}$$
$$\times \sum_{j,i=0}^{\infty} \upsilon_{j,i}\gamma\left(1-\frac{r}{\beta}, (aj+a+i)\left(\frac{\delta}{t}\right)^\beta\right).$$

The mean inactivity time (MIT), also called mean reversed residual life (MRRL) function, is defined by $M_1(t) = E[(t-X)|X \le t]$. It represents the waiting time elapsed since the failure of an itemon condition that this failure had occurred in $(0, x)$. The MIT of the Kw-MOFr distribution can be determined by setting $n = 1$ in the last equation.

5. ORDER STATISTICS AND MOMENTS

Let $X_1, \dots, X_n$ be a random sample of size n from the Kw-MOFr model with cdf and pdf as given in (5) and (6), respectively. Let $X_{1:n}, \dots, X_{n:n}$ be the corresponding order statistics. Then, the pdf of $X_{j:n}, 1 \le j \le n$, denoted by $f_{j:n}(x)$, is given by

$$f_{j:n}(x) = C_{j:n}\alpha\beta ab\delta^{\beta}x^{-\beta-1}l_x^{-(a+1)}\left\{1-\exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]l_x^{-a}\right\}^{b(n-j+1)-1}$$
$$\times \exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left\{1-\left(1-\exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]l_x^{-a}\right)^{b}\right\}^{j-1},$$

where $C_{j:n} = n!/[(j-1)!\,(n-j)!]$.

The joint pdf of $X_{i:n}$ and $X_{j:n}$, for $1 \le i \le j \le n$, is given (for $0 \le x \le y \le \infty$) by

$$f_{i:j:n}(x,y) = C_{i:j:n}\, f(x)f(y)F(x)^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(x)]^{n-j}.$$

Then, by inserting (5) and (6) in the last equation, we obtain

$$f_{i:j:n}(x,y) = C_{i:j:n}\alpha\beta ab\delta^{\beta}(xy)^{-\beta-1}\exp(b_x)\exp(b_y)(l_xl_y)^{-a-1}$$
$$\times\{[1-\exp(b_x)l_x^{-a}][1-\exp(b_y)l_y^{-a}]\}^{b-1}$$
$$\times\left\{[1-\exp(b_x)l_x^{-a}]^{b}-[1-\exp(b_y)l_y^{-a}]^{b}\right\}^{j-i-1}$$
$$\times\{1-[1-\exp(b_x)l_x^{-a}]^{b}\}^{i-1}\{[1-\exp(b_x)l_x^{-a}]^{b}\}^{n-j},$$

where $C_{i:j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$, $l_t = \left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\delta}{t}\right)^{\beta}\right]\right\}$ and $b_t = -a\left(\frac{\delta}{t}\right)^{\beta}$ for $t = x, y$.

The pdf of $X_{j:n}$ can also be expressed by

$$f_{j:n}(x) = \frac{f(x)}{\mathrm{B}(j,n-j+1)}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}F^{j+r-1}(x). \quad (12)$$

Therefore, we can write

$$F^{j+r-1}(x) = \sum_{s=0}^{\infty}(-1)^k\binom{j+r-1}{s}$$
$$\times\left\{1-\exp\left[-a\left(\frac{\delta}{x}\right)^{\beta}\right]\left(\alpha+(1-\alpha)\exp\left[-\left(\frac{\delta}{x}\right)^{\beta}\right]\right)^{-a}\right\}^{bs}.$$

By inserting (5) and the last equation in (12), we obtain

$$f_{j:n}(x) = \sum_{l,k=0}^{\infty}B_{l,k}h_{a(l+1)+k}(x), \quad (13)$$

where $h_{a(l+1)+k}(x)$is the Fr pdf with parameters β and $\delta[a(l+1)+k]^{1/\beta}$ and

$$B_{l,k} = \frac{ab}{\mathrm{B}(i, n-i+1)} \sum_{s=0}^{\infty} \sum_{r=0}^{n-1} \frac{(-1)^{s+l+r}\Gamma(al+a+k)}{k!\,\Gamma(al+a+1)} \times \binom{n-1}{r} \left(1-\frac{1}{\alpha}\right)^h \binom{j+r-1}{s} \binom{b(s+1)}{l}.$$

So, the density function of the Kw-MOFr order statistics is a linear combination of Fr densities. Based on equation (13), we can obtain some structural properties of $Y_{j:n}$ from those Fr properties.

The qth moment of $X_{j:n}$ is given by

$$E\left(X_{j:n}^q\right) = \sum_{l,k=0}^{\infty} B_{l,k} E\left(Y_{a(l+1)+k}^q\right). \tag{14}$$

Based upon the moments (14), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of suitable Fr expected values given by

$$\lambda_k = \frac{1}{k} \sum_{d=0}^{k-1} (-1)^d \binom{k-1}{d} E(X_{k-d:d}), k \geq 1.$$

6. ESTIMATION

Several approaches for parameter estimation were proposed in the statistical literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties for constructing confidence intervals. We investigate the estimation of the parameters of the Kw-MOFr distribution by maximum likelihood for complete data sets. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a random sample of this distribution with unknown parameter vector $\theta = (\alpha, \beta, \delta, a, b)^T$. Then, the log-likelihood function for θ, say $\ell = \ell(\theta)$, is

$$\ell(\theta) = n\log\left(\beta\, a\, b\, \delta^\beta\right) - (\beta+1)\sum_{i=1}^n \log x_i - a\sum_{i=1}^n \left(\frac{\delta}{x_i}\right)^\beta$$
$$-(a+1)\sum_{i=1}^n \log(z_i) + (b-1)\sum_{i=1}^n \log[1 - z_i^{-a}(1-s_i)^a],$$

where $z_i = \alpha + (1-\alpha)(1-s_i)$ and $s_i = 1 - \exp\left[-\left(\frac{\delta}{x_i}\right)^\beta\right]$.

The MLE $\hat{\theta}$ of θ can be determined by maximizing $\ell(\theta)$ (for a given **x**) either directly by using the Mathcad, R (optim function), SAS (PROC NLMIXED), Ox program (sub-routine MaxBFGS) or by solving the nonlinear system obtained by differentiating this equation and equating its five components to zero.

Let $p_i = \left(\frac{\delta}{x_i}\right)^\beta (1-s_i)^a \log\left(\frac{\delta}{x_i}\right)$ and $t_i = (1-s_i)\left(\frac{\beta}{\delta}\right)\left(\frac{\delta}{x_i}\right)^\beta$.
The components of the score vector $\mathbf{U}(\theta) = \frac{\partial\ell}{\partial\theta} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b})^T$ are given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - (a+1)\sum_{i=1}^n \frac{s_i}{z_i} + (b-1)\sum_{i=1}^n \frac{a s_i z_i^{-(a+1)}(1-s_i)^a}{1-z_i^{-a}(1-s_i)^a},$$
$$\frac{\partial\ell}{\partial\beta} = n\left(\frac{1}{\beta} + \log\delta\right) - \sum_{i=1}^n \log x_i + (a+1)\sum_{i=1}^n (1-\alpha)(1-s_i)^{a-1}$$
$$+a(b-1)\sum_{i=1}^n \frac{p_i[1-(1-\alpha)z_i^{-1}(1-s_i)^a]}{z_i^\alpha - (1-s_i)^a} - a\sum_{i=1}^n p_i(1-s_i)^a,$$

$$\frac{\partial \ell}{\partial \delta}=\frac{n\beta}{\delta}-a\sum_{i=1}^{n}t_i(1-s_i)^{-1}+(a+1)\sum_{i=1}^{n}\frac{(1-\alpha)t_i}{z_i}$$
$$+a(b-1)\sum_{i=1}^{n}\frac{t_i(1-s_i)^{a-1}}{z_i^{a}-(1-s_i)^{a}}\left[1-(1-\alpha)z_i^{-1}(1-s_i)\right],$$
$$\frac{\partial \ell}{\partial a}=\frac{n}{a}-\sum_{i=1}^{n}\left(\frac{\delta}{x_i}\right)^{\beta}-\sum_{i=1}^{n}\log z_i+(b-1)\sum_{i=1}^{n}\frac{p_i/\log(\frac{\delta}{x_i})+(1-s_i)^{a}\log z_i}{z_i^{a}-(1-s_i)^{a}}$$

and

$$\frac{\partial \ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log[1-z_i^{-a}(1-s_i)^{a}].$$

It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function. For interval estimation of the model parameters, we require the 5×5 observed information matrix $J(\theta)=\{J_{rs}\}$ (for $r,s=\alpha,\beta,\delta,a,b$). Under standard regularity conditions, the multivariate normal $N_5(0,J(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\hat{\theta})$ is the total observed information matrix evaluated at $\hat{\theta}$. Therefore, approximate $100(1-\phi)\%$ confidence intervals for α,β,δ,a and b can be determined as:

$\hat{\alpha}\pm z_{\frac{\phi}{2}}\sqrt{\hat{J}_{\alpha\alpha}}$, $\hat{\beta}\pm z_{\frac{\phi}{2}}\sqrt{\hat{J}_{\beta\beta}}$, $\hat{\delta}\pm z_{\frac{\phi}{2}}\sqrt{\hat{J}_{\delta\delta}}$, $\hat{a}\pm z_{\frac{\phi}{2}}\sqrt{\hat{J}_{aa}}$ and $\hat{b}\pm z_{\frac{\phi}{2}}\sqrt{\hat{J}_{bb}}$, where $z_{\frac{\phi}{2}}$ is the upper ϕth percentile of the standard normal distribution.

7. DATA ANALYSIS

In this section, we provide applications of the Kw-MOFr distribution using two real data sets. We compare the proposed distribution with other related lifetime models, namely: the MOFr, Kw-Fr, beta Fr (BFr), MOIE, MOIR, EFr and Fr distributions. The pdf of the BFr model is given (for $x>0$) by

$$f(x)=\frac{\beta\delta^{\beta}}{B(a,b)}\,x^{-\beta-1}\,e^{-a\left(\frac{\delta}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\delta}{x}\right)^{\beta}}\right]^{b-1},$$

where β,δ,a and b are positive parameters.

In order to compare the distributions, we consider some criteria like the maximized log-likelihood ($-2\hat{\ell}$), Akaike information criterion (AIC), consistent Akaike information criterion ($CAIC$), Bayesian information criterion (BIC) and Hannan-Quinn information criterion ($HQIC$). The model with minimum values for AIC or BIC or $HQIC$ or $CAIC$ can be chosen as the best model to fit the data.

The first data set from Bjerkedal (1960) consists of 72 survival times for guinea pigs injected with different doses of tubercle bacilli: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376. These data were previously studied by Krishna et al. (2013).

The values of $-2\hat{\ell}$, AIC, BIC, $HQIC$ and $CAIC$ for the fitted models are given in Tables 2 and 4. The MLEs and their corresponding standard errors (in parentheses) of the model parameters for the fitted Kw-MOIE, Kw-MOIR, Kw-Fr, MOFr, BFr, EFr and Fr distributions to both data sets are listed in Tables 3 and 5. The figures in these tables are obtained using the Mathcad program. The histogram and the estimated pdfs and cdfs for the best fitted models to the survival time guinea pigs and strengths of 1.5 cm glass fibres are displayed in Figures 4 and 5, respectively.

Table 2
The $-2\hat{\ell}, AIC, BIC, HQIC$ and $CAIC$ Statistics for the First Data Set

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$
Kw-MOFr	741.6	751.6	762.9	756.1	752.5
EFr	780.5	786.5	793.3	789.2	786.9
Kw-Fr	780.5	788.5	797.6	792.1	789.1
BFr	780.6	788.6	797.7	792.3	789.2
Kw-MOIE	782.7	790.7	799.8	794.3	791.3
Fr	791.3	795.3	799.9	797.1	795.5
MOFr	790.1	796.1	802.9	798.8	796.5
Kw-MOIR	800.2	808.2	817.3	811.8	808.8

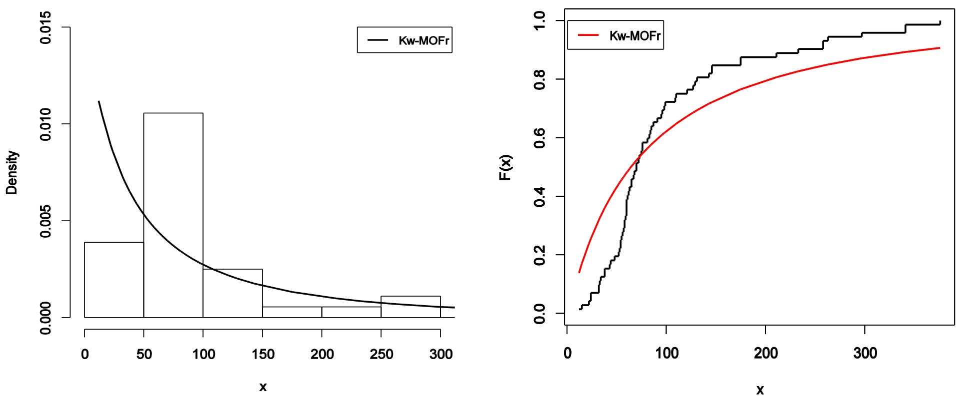


Figure 4: Fitted pdfs and cdfs to the Survival Times for Guinea Pigs

The second data set obtained from Smith and Naylor (1987) consists of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The 63 observations are: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 , 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1. 61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89. These data were previously studied by Barreto-Souza et al. (2011) and Afify et al. (2015).

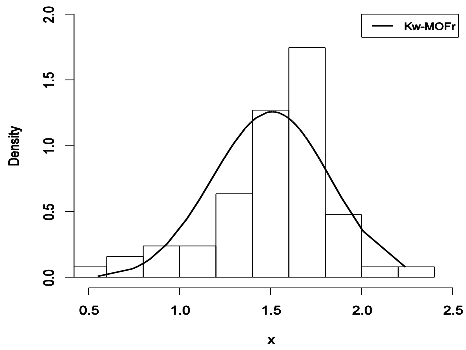
Table 3
MLEs and their Standard Errors for the First Data Set

Model	Estimates				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{a}$	$\hat{b}$
Kw-MOFr	0.068	0.087	69.693	54.638	308.470
	(0.984)	(0.038)	(0.082)	(0.063)	(0.229)
EFr	–	0.6207	336.3679	–	8.2723
	–	(0.208)	(374.803)	–	(7.953)
Kw-Fr	–	0.6207	0.7111	45.7326	8.2723
	–	(0.003)	(0.013)	(0.092)	(0.979)
BFr	–	0.322	24.5032	19.9786	20.1331
	–	(0.00115)	(0.087)	(7.246)	(7.26)
Kw-MOIE	8.8727	–	0.1758	68.1393	2.6258
	(1.174)	–	(0.000)	(0.020)	(0.512)
Fr	–	1.4148	54.1888	–	–
	–	(0.00271)	(0.111)	–	–
MOFr	14.9816	1.7855	13.991	–	–
	(4.6305)	(0.193)	(2.964)	–	–
Kw-MOIR	9.993	–	1.6788	58.4697	0.6389
	(1.972)	–	(0.001)	(0.105)	(0.098)

Table 4
The $-2\hat{\ell}, AIC, BIC, HQIC$ and $CAIC$ Statistics for the Second Data Set

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$
Kw-MOFr	36.5	46.5	57.2	50.7	47.6
Kw-Fr	39.6	47.6	56.2	51	48.3
Kw-MOIE	41.6	49.6	58.2	52.9	50.3
EFr	44.3	50.5	56.7	52.8	50.7
BFr	61.7	69.7	78.3	73.1	70.4
Kw-MOIR	67.3	75.3	83.9	78.7	76
Fr	93.7	97.7	102	99.4	97.9
MOF	95.7	101.7	108.2	104.2	102.1

Based on the figures in Tables 2 and 4, we can compare the Kw-MOFr model with the other fitted models. It is clear from these figures that the Kw-MOFr model gives the lowest values for the AIC, BIC, $HQIC$ and $CAIC$ statistics (except BIC for the Kw-Fr model in the second data set) among all fitted models. The plots in Figures 3 and 4 also indicate the same thing. So, the Kw-MOFr model could be chosen as the best model.



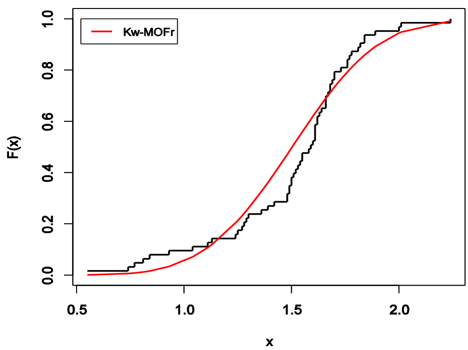


Figure 5: Fitted pdfs and cdfs to the Strengths of Glass Fibres

Table 5
MLEs and their Standard Errors for the Second Data Set

Model	**Estimates**				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{a}$	$\hat{b}$
Kw-MOFr	20.8799	0.7699	0.0059	30.8587	2031.5918
	(6.93)	(0.079)	(0.002)	(0.034)	(9.573)
Kw-Fr	–	0.740	2.116	5.504	857.343
	–	(0.071)	(4.555)	(7.982)	(153.948)
Kw-MOIE	5.43	0.0898	–	18.7573	163.3153
	(0.152)	(0.00013)	–	(0.025)	(4.542)
EFr	–	0.999	7.816	–	132.827
	–	(0.136)	(2.945)	–	(116.63)
BFr	–	0.685	1.331	19.591	30.411
	–	(0.181)	(1.085)	(18.115)	(18.238)
Kw-MOIR	1	–	2.7498	0.5971	5.7974
	(0.192)	–	(0.079)	(0.034)	(0.00819)
Fr	–	2.888	1.264	–	–
	–	(0.234)	(0.059)	–	–
MOFr	0.4816	2.3876	1.5441	–	–
	(0.252)	(0.253)	(0.226)	–	–

8. CONCLUSIONS

We propose a new five-parameter lifetime model, called the Kumaraswamy Marshall–Olkin Fréchet (Kw-MOFr) distribution, which extends the Marshall–Olkin Fréchet (MOFr) (Krishna et al., 2013) distribution. We prove that the Kw-MOFr density function can be given as a mixture of Fréchet densities. We obtain explicit expressions for the ordinary and incomplete moments, quantile and generating functions, mean deviations, moments of the residual life, moments of the reversed residual life and Rényi and δ-entropies. We also investigate the density function of order statistics and their moments. We discuss the maximum likelihood estimation of the model parameters. We prove empirically that the Kw-MOFr model can give a better fit than other lifetime models by means of two real data sets.

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