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نموذج اجابة نصف ورقة

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الأســـئلة:

Solve the following (three) questions.

First Question:

1) *Define: least upper bound, greatest lower bound, countable set, convergent sequence, bounded sequence, Cauchy sequence, continuous function.*
2) *Show that if E and F are bounded subsets of R and $E \subseteq F$, then*
$$\inf F \le \inf E \le \sup E \le \sup F.$$
3) *Prove that every convergent sequence is bounded.*
4) *Every convergent sequence is a Cauchy sequence.*
5) *Show that any constant function $f(x) = c$, is integrable on any interval $[a,b]$.*

Second Question:

1) *Prove that a uniform limit of a sequence of continuous functions is also continuous.*

2) *Define: upper Darbaux sum, lower Darboux sum, Riemann integrable function.*

3) *Show that for a bounded function* $f(x)$ *on an interval [a,b], if the partition* P^* *is finer than the partition* P*, then*

$$L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P).$$

4) *Study the convergence of he sequence* $\{f_n(x)\} = \left\{\frac{1}{1+nx}\right\}$ *of functions on the interval [0,1].*

Solutions

First Question:

1) *least upper bound*

A least upper bound of a subset $E \subseteq R$ *is a real number* α*, called* $\sup E = \alpha$*, if*

1- $\alpha \ge x$ *for all* $x \in E$

2- *for all* $\varepsilon > 0 \ \exists\ x_o$ *s. t.* $x_o > \alpha - \varepsilon$.

greatest lower bound,

A greatest lower bound of a subset $E \subseteq R$ *is a real number* β*, called* $\inf E = \beta$*, if*

1- $\beta \le x$ *for all* $x \in E$

2- *for all* $\varepsilon > 0 \ \exists\ x_o$ *s. t.* $x_o < \beta + \varepsilon$

countable set,

A set E is called a countable set if there exists a one to one corresponding between E and the set of natural numbers N.

convergent sequence,

A sequence $\{x_n\}$ *is called a convergent sequence if there exist a real number* $x_o \in R$ *such that* $\lim_{n\to\infty} x_n = x_o$ *i.e.*

$\forall\ \varepsilon > 0\ \exists\ n_o \in N$ *such that* $n \ge n_o \Rightarrow |x_n - x_o| < \varepsilon.$

bounded sequence,

A sequence $\{x_n\}$ *is called a bounded sequence if there exist a real number* $M \in R$ *such that* $|x_n| \le M, \quad \forall\, n \in N$

Cauchy sequence,

A sequence $\{x_n\}$ *is called a Cauchy sequence if.*

$$\forall\, \varepsilon > 0 \;\; \exists\, n_o \in N \quad \textit{such that} \;\; |x_n - x_m| < \varepsilon \quad \forall n,m \ge n_o.$$

continuous function.

A real valued function $f(x)$ *is continuous at a point* x_o *if*

$\lim_{x \to x_o} f(x) = f(x_o)$ *i.e.*

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0 \quad \textit{such that} \;\; |x - x_o| < \delta \Rightarrow |f(x) - f(x_o)| < \varepsilon.$$

2) *if* E *and* F *are bounded subsets of* R *and* $E \subseteq F$ *, then*

- $\inf\ F = \beta_F \Rightarrow \beta_F \le x \;\; \forall\, x \in F$

$E \subseteq F \Rightarrow \beta_F \le x \;\; \forall\, x \in E$

$\Rightarrow \beta_F$ *is a lower bound of* E

$\Rightarrow \beta_F \le \inf\ E$ ----------------------(1)

- *We know that for a bounded set E:*

  $\inf\ E \le \sup\ E$ ----------------------(2)

- $\sup\ F = \alpha_F \Rightarrow \alpha_F \ge x \;\; \forall\, x \in F$

$E \subseteq F \Rightarrow \alpha_F \ge x \;\; \forall\, x \in E$

$\Rightarrow \alpha_F$ *is an upper bound of* E

$\Rightarrow \alpha_F \ge \sup\ E$ ----------------------(2)

From (1), (2) and (3), we get

$$\inf F \le \inf E \le \sup E \le \sup F. \qquad ***$$

3)

• *let* $\{x_n\}$ *be a convergent sequence to a real number* x_o*, then*

$\forall\ \varepsilon > 0\ \ \exists\ n_o \in N$ *such that* $|x_n - x_o| < \varepsilon\ \ \forall\ n \geq n_o.$

for $\varepsilon = 1 \Rightarrow |x_n - x_o| < 1\ \ \forall\ n \geq n_o$

$\Rightarrow |x_n| - |x_o| \leq |x_n - x_o| < 1\quad \forall\ n \geq n_o$

$\Rightarrow |x_n| \leq 1 + |x_o|\quad \forall\ n \geq n_o$ ------------------------------(1)

• *Its clear that*

$|x_n| \leq \max\{|x_i|,\ i = 1,2,3,\dots,n_o\text{-}1\}\ \ \forall\ n < n_o$ --------------------(2)

(1) *and* (2) $\Rightarrow |x_n| \leq \{\max\{|x_i|,\ i = 1,2,3,\dots,n_o\text{-}1\}, 1 + |x_o|\}\ \ \forall\ n \in N$

$\Rightarrow \{x_n\}$ *be a bouunded sequence.*

4)

Let $\{x_n\}$ be a convergent sequence to areal number x_o, then

$\Rightarrow \forall\ \varepsilon > 0\ \ \exists\ n_o \in N$ *such that* $|x_n - x_o| < \frac{\varepsilon}{2}\quad \forall\ n \geq n_o.$

$\&\ \forall\ \varepsilon > 0\ \ \exists\ n_o \in N$ *such that* $|x_m - x_o| < \frac{\varepsilon}{2}\quad \forall\ m \geq n_o.$

$$\Rightarrow |x_n - x_m| = |x_n - x_o + x_o - x_m|$$
$$\leq |x_n - x_o| + |x_m - x_o|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence.

5) *Under any partition* $P = \{a = x_o < x_1 < \dots < x_n = b\}$ *of any interval* $[a,b]$*, we have*

$$M_i = \sup\{f(x): x_{i-1} \le x \le x_i\} = c$$
$$m_i = \inf\{f(x): x_{i-1} \le x \le x_i\} = c$$
$$\Rightarrow \begin{cases} U(f.P) = \sum_{i=1}^{n} M_i \Delta x_i = c(b-a) \\ L(f.P) = \sum_{i=1}^{n} m_i \Delta x_i = c(b-a) \end{cases}$$
$$\Rightarrow U(f.P) = L(f.P) = c(b-a)$$
$\Rightarrow$ $f(x)$ is Riemann integrable function
$$\Rightarrow R\int_a^b f(x)dx = c(b-a)$$

Second Question

1) Let $\{f_n(x)\}$ be a sequence of continuous functions on the same domain D which converges uniformaly on D to a function $f(x)$.

Let $x_o \in D$ be arbitrary.

$$\because f_n(x) \xrightarrow{uniformaly} f(x) \quad as \ \ n \to \infty)$$
$$\Rightarrow \forall \ \ \varepsilon > 0 \ \ \exists n_o(\varepsilon \varepsilon \in N \ \ s.\ t.\ n \ge n_o \ \Rightarrow \ |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in D \ \text{---------------}(1)$$
$$\Rightarrow \forall \ \ \varepsilon > 0 \quad \Rightarrow \ |f_{n_o}(x_o) - f(x_o)| < \frac{\varepsilon}{3} \quad \text{------------------ -- ---------------}(2)$$
$$\because f_{n_o}(x) \ is\ continuous\ at\ the\ point\ x_o$$
$$\Rightarrow \forall \ \ \varepsilon > 0 \ \ \exists \delta_{n_o}(\varepsilon) > 0 \ \ s.\ t.\ |x - x_o| \ < \delta \Rightarrow \ |f_{n_o}(x) - f_{n_o}(x_o)| < \frac{\varepsilon}{3} \quad \text{--------- ------}(3)$$

(1),(2) & (3), we have

$$\forall \ \ \varepsilon > 0 \ \ \exists \delta_{n_o}(\varepsilon) > 0 \ \ s.\ t.\ |x - x_o| \ < \delta$$
$$\Rightarrow |f(x) - f(x_o)| = |f(x) - f_{n_o}(x) + f_{n_o}(x) - f_{n_o}(x_o) + f_{n_o}(x_o) - f(x_o)|$$
$$\prec \ |f(x) - f_{n_o}(x)| + |f_{n_o}(x) - f_{n_o}(x_o)| + |f_{n_o}(x_o) - f(x_o)|$$
$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon$$

Hence, $f(x)$ is continous at any point of the domain D.

2) *upper Darbaux sum, lower Darboux sum,*

Let $f(x)$ *be a bounded real valued function on [a,b]. Under any partition* $P = \{a = x_o < x_1 < \ldots < x_n = b\}$ *of an interval [a,b], we define:*

$$M_i := \sup\{f(x) : x_{i-1} \le x \le x_i\}$$
$$m_i := \inf\{f(x) : x_{i-1} \le x \le x_i\}$$
$$\Rightarrow \begin{cases} U(f.P) := \sum_{i=1}^{n} M_i \Delta x_i \\ L(f.P) = \sum_{i=1}^{n} m_i \Delta x_i \end{cases}$$

U(f,p):upper Darbaux sum

L(f,P): lower Darbaux sum

Riemann integrable function.

Let $f(x)$ *be a bounded real valued function on [a,b]. A function* $f(x)$ *is called Riemann intagrable if*

$$\inf_P U(f.P) = \sup_P L(f.P)$$
$$\Rightarrow R\int_a^b f(x)dx = \inf_P U(f.P)$$

3) *Without loss of generality we may consider that the partition* P^*, *finer than the partition* P, *has exactly one point* x^* *more than* P.

$Let\ x_{i-1} < x^* < x_i$

- $U(f.P^*) \le U(f.P)$?

$Let\ M_i^* := \sup\{f(x) : x_{i-1} \le x \le x^*\}$

$M_i^{**} := \sup\{f(x) : x^* \le x \le x_i\}$

$\Rightarrow M_i^* \le M_i \ \& \ M_i^{**} \le M_i$

$\Rightarrow U(f.P) - U(f.P^*) = M_i(x_i - x_{i-1}) - M_i^*(x^* - x_{i-1}) - M_i^{**}(x_i - x^*)$

$= (M_i\text{-}M_i^*)(x^* - x_{i-1}) + (M_i\text{-}M_i^{**})(x_i - x^*) \ge 0$

$\Rightarrow U(f.P) \ge U(f.P^*)$ --------------------------------(1)

- Similarly, we can verify that

$L(f.P) \le L(f.P^*)$ --------------------------------(2)

$(1)\ \&\ (2) \Rightarrow L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P).$

4) The sequence $\{f_n(x)\} = \left\{\dfrac{1}{1+nx}\right\}$ of functions on the interval [0,1] is pint-wise convergence sequence.

- $For\ x=0 \ \Rightarrow \{f_n(x)\} = \left\{\dfrac{1}{1+nx}\right\} = \{1\}$ ------------(1)

- $For\ 0 < x \le 1 \ \Rightarrow \lim\limits_{n\to\infty}\{f_n(x)\} = \lim\limits_{n\to\infty}\left\{\dfrac{1}{1+nx}\right\} = 0$ ------(2)

$(1)\&(2) \Rightarrow \lim\limits_{n\to\infty}\{f_n(x)\} = f(x) := \begin{cases} 1 & for\ x=0 \\ 0 & for\ 0 < x \le 1 \end{cases}$

$\Rightarrow \{f_n(x)\}$ converges pointwise to $f(x)$

- $f(x)$ is not continuous and by theorem

$\Rightarrow$ the convergence of $\{f_n(x)\}$ is not uniform convergence to $f(x)$ ****

انتهت الإجابة

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