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نموذج اجابة ورقة كاملة

المادة: **تحليل حقيقي كود ٢٢٣ر**

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الأسئلة:

Solve the following (three) questions.

First Question:

Decide whether each of the following statements is true or false. Justify your answer (by providing a proof, a counter example, a correction, an interpretation):

1) *Countable union of countable sets is also countable*
2) *The sum of two convergent sequences is also convergent sequence.*
3) *Every bounded sequence is convergent.*
4) *The rational number system Q is complete.*
5) *Any constant function $f(x) = c$, is integrable on any interval $[a,b]$.*
6) *The uniform convergence of a sequence of functions $\{f_n(x)\}$ is stronger than its pointwise convergence.*

Second Question:

1) *Define: least upper bound, greatest lower bound, countable set, convergent sequence, bounded sequence, Cauchy sequence, continuous function.*
2) *Let $\{a_i\}$ and $\{b_i\}$, $i=1,2,\dots,n$ be real numbers. Then,*

   $\sup(a_i + b_i) \le \sup(a_i) + \sup(b_i)$.

3) *Every convergent sequence is a Cauchy sequence.*
4) *Prove that every convergent sequence is bounded.*

Third Question:

1) *Prove that a uniform limit of a sequence of continuous functions is also continuous.*
2) *Define: upper Darbaux sum, lower Darboux sum, Riemann integrable function.*
3) *Show that for a bounded function* $f(x)$ *on an interval [a,b], if the partition* P^* *is finer than the partition* P *, then*
$$L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P).$$
4) *Study the convergence of he sequence* $\{f_n(x)\} = \left\{\frac{1}{1+nx}\right\}$ *of functions on the interval [0,1].*

Solutions

First Question:

1) *Countable union of countable sets is also countable (True):*

Let $\{X_n\}$ *be accountable collection of countable sets, then each of these can be written as a sequence:*

$$X_1 := \{x_{11}, x_{12}, x_{13}, x_{14}, \dots\dots\}$$
$$X_2 := \{x_{21}, x_{22}, x_{23}, x_{24}, \dots\dots\}$$
$$X_3 := \{x_{31}, x_{32}, x_{33}, x_{34}, \dots\dots\}$$
$$X_4 := \{x_{41}, x_{42}, x_{43}, x_{44}, \dots\dots\}$$
$$\vdots$$
$$X_n := \{x_{n1}, x_{n2}, x_{n3}, x_{n4}, \dots\dots\}$$
$$\vdots$$
$$\Rightarrow X := \bigcup_{n\in N} X_n = \{x_{11}, x_{21}, x_{12}, x_{13}, x_{22}, x_{31}, x_{41}, x_{32}, x_{23}, x_{14} \dots\dots\}$$
$$\Rightarrow X := \bigcup_{n\in N} X_n \text{ written as a sequence.}$$
$$\Rightarrow X := \bigcup_{n\in N} X_n \text{ is countable set.}$$

2) *The sum of two convergent sequences is also convergent sequence. (True):*

Let $\lim_{n\to\infty} x_n = x_o$ and $\lim_{n\to\infty} y_n = y_o$, then

$$\Rightarrow \forall\, \varepsilon > 0 \;\; \exists\, n_1 \in N \quad \text{such that} \quad |x_n - x_o| < \frac{\varepsilon}{2} \quad \forall\, n \geq n_1$$
$$\& \;\; \forall\, \varepsilon > 0 \;\; \exists\, n_2 \in N \quad \text{such that} \quad |y_n - y_o| < \frac{\varepsilon}{2} \quad \forall\, n \geq n_2$$
$$\text{For } n_o := \max\{n_1, n_2\} \Rightarrow \begin{cases} |x_n - x_o| < \frac{\varepsilon}{2} & \forall\, n \geq n_o \\ |y_n - y_o| < \frac{\varepsilon}{2} & \forall\, n \geq n_o \end{cases}$$
$$\Rightarrow |(x_n + y_n) - (x_o + y_o)| \leq |x_n - x_o| + |y_n - y_o|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall\, n \geq n_o$$
$$\Rightarrow |(x_n + y_n) - (x_o + y_o)| < \varepsilon \qquad \forall\, n \geq n_o$$
$$\Rightarrow \{x_n + y_n\} \text{ is a convergence sequence.}$$

3) *Every bounded sequence is convergent.* *(False:)*

Since $\{(-1)^n\} = \{1, -1, 1, -1, 1, -1, \dots\}$ is a bounded sequence ($|x_n| \leq 1 \;\; \forall\, n \in N$)

but is not a convergent seq. ($\lim_{n\to\infty}(-1)^n = \begin{cases} 1 & \text{if n even} \\ -1 & \text{if n odd} \end{cases}$).

4) *The rational number system Q is complete.* (False:)

Since the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ *is a Cauchy seq. in* Q, *but is not convergent seq. to a point in* Q $[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \notin Q\,]$.

5) *Any constant function* $f(x)=c$, *is integrable on any interval [a,b].* (True:)

Under any partition $P=\{a=x_o<x_1<\ldots<x_n=b\}$ *of any interval [a,b], we have*

$$M_i=\sup\{f(x):x_{i-1}\le x\le x_i\}=c$$

$$m_i=\inf\{f(x):x_{i-1}\le x\le x_i\}=c$$

$$\Rightarrow\begin{cases} U(f.P)=\sum_{i=1}^{n}M_i\Delta x_i=c(b-a)\\ L(f.P)=\sum_{i=1}^{n}m_i\Delta x_i=c(b-a)\end{cases}$$

$$\Rightarrow U(f.P)=L(f.P)=c(b-a)$$

$$\Rightarrow f(x)\text{ is Riemann integrable function}$$

$$\Rightarrow R\int_a^b f(x)dx=c(b-a)$$

6) *The uniform convergence of a sequence of functions* $\{f_n(x)\}$ *is stronger than its pointwise convergence.* (True:)

Since the uniform convergence of a sequence of functions implies its pointwise convergence.

Second Question:

1) least upper bound

A least upper bound of a subset $E\subseteq R$ *is a real number* α, *called* $\sup E=\alpha$, *if*

1- $\alpha\ge x$ *for all* $x\in E$

2- *for all* $\varepsilon>0\ \exists\, x_o$ *s. t.* $x_o>\alpha-\varepsilon$.

greatest lower bound,

A greatest lower bound of a subset $E \subseteq R$ is a real number β, called $\inf E = \beta$, if

1- $\beta \leq x$ for all $x \in E$

2- for all $\varepsilon > 0$ $\exists\, x_o$ s. t. $x_o < \beta + \varepsilon$

countable set,

A set E is called a countable set if there exists a one to one corresponding between E and the set of natural numbers N.

convergent sequence,

A sequence $\{x_n\}$ is called a convergent sequence if there exist a real number $x_o \in R$ such that $\lim_{n \to \infty} x_n = x_o$ i.e.

$\forall\, \varepsilon > 0\ \exists\, n_o \in N$ such that $n \geq n_o \Rightarrow |x_n - x_o| < \varepsilon.$

bounded sequence,

A sequence $\{x_n\}$ is called a bounded sequence if there exist a real number $M \in R$ such that $|x_n| \leq M, \quad \forall\, n \in N$

Cauchy sequence,

A sequence $\{x_n\}$ is called a Cauchy sequence if.

$\forall\, \varepsilon > 0\ \exists\, n_o \in N$ such that $|x_n - x_m| < \varepsilon \quad \forall n,m \geq n_o.$

continuous function.

A real valued function $f(x)$ is continuous at a point x_o if

$\lim_{x \to x_o} f(x) = f(x_o)$ i.e.

$\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ such that $|x - x_o| < \delta \Rightarrow |f(x) - f(x_o)| < \varepsilon.$

2) Let $\{a_i\}$ and $\{b_i\}$, i=1,2,…,n be real numbers. Then,

$\sup(a_i + b_i) \leq \sup(a_i) + \sup(b_i)$. (True):

$$Let\ \sup\{a_i\} = \alpha \ \ and \ \ \sup\{b_i\} = \beta\ , then$$

$$\alpha \geq a_i \quad and \quad \beta \geq b_i \quad for\ \ i = 1,2,\ldots,n$$

$$\Rightarrow \alpha + \beta \geq a_i + b_i \quad for\ \ i = 1,2,\ldots,n$$

$$\Rightarrow \alpha + \beta \ is\ an\ upper\ bound\ of\ \{a_i + b_i\}$$

$$\Rightarrow \alpha + \beta \geq \sup\{a_i + b_i\}$$

$$\Rightarrow \sup(a_i + b_i) \leq \sup(a_i) + \sup(b_i) \quad *****$$

3)

Let $\{x_n\}$ be a convergent sequence to areal number x_o, then

$$\Rightarrow \forall\ \varepsilon > 0\ \ \exists\ n_o \in N \quad such\ that\ \ |x_n - x_o| < \frac{\varepsilon}{2} \quad \forall\ n \geq n_o.$$

$$\&\ \forall\ \varepsilon > 0\ \ \exists\ n_o \in N \quad such\ that\ \ |x_m - x_o| < \frac{\varepsilon}{2} \quad \forall\ m \geq n_o.$$

$$\Rightarrow |x_n - x_m| = |x_n - x_o + x_o - x_m|$$

$$\leq |x_n - x_o| + |x_m - x_o|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence.

4)

- $let\ \{x_n\}\ be\ a\ con\,vergent\ se\,quence\ to\ \ a\ real\ num\,ber\ x_o,\ then$

$$\forall\ \varepsilon > 0\ \ \exists\ n_o \in N \quad such\ that\ \ |x_n - x_o| < \varepsilon \quad \forall\ n \geq n_o.$$

$$for\ \ \varepsilon = 1 \Rightarrow |x_n - x_o| < 1 \quad \forall\ n \geq n_o$$

$$\Rightarrow |x_n| - |x_o| \leq |x_n - x_o| < 1 \quad \forall\ n \geq n_o$$

$$\Rightarrow |x_n| \leq 1 + |x_o| \quad \forall\ n \geq n_o \qquad \text{-----------------------------}(1)$$

- $Its\ clear\ that$

$$|x_n| \leq \max\{|x_i|,\ i = 1,2,3,\ldots,n_o\text{-}1\ \} \quad \forall\ n < n_o \qquad \text{--------------------}(2)$$

$$(1)\ and\ (2) \Rightarrow |x_n| \leq \{\max\{|x_i|,\ i = 1,2,3,\ldots,n_o\text{-}1\ \},1 + |x_o|\ \} \quad \forall\ n \in N$$

$$\Rightarrow \{x_n\}\ be\ a\ bou\,unded\ sequ\,ence.$$

Third Question

1) Let $\{f_n(x)\}$ be a sequence of continuous functions on the same domain D which converges uniformaly on D to a function $f(x)$.

Let $x_o \in D$ *be arbitrary.*

$\because\ f_n(x) \xrightarrow{uniformaly} f(x) \quad as\ \ n \to \infty)$

$\Rightarrow \forall\ \ \varepsilon > 0\ \ \exists n_o(\varepsilon\varepsilon \in N\ \ s.\ t.\ n \geq n_o\ \ \Rightarrow\ \ |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in D$ ---------------(1)

$\Rightarrow \forall\ \ \varepsilon > 0 \quad \Rightarrow\ \ |f_{n_o}(x_o) - f(x_o)| < \frac{\varepsilon}{3}$ ------------------ -- ---------------(2)

$\because\ f_{n_o}(x)$ *is continuous at the point* x_o

$\Rightarrow \forall\ \ \varepsilon > 0\ \ \exists\delta_{n_o}(\varepsilon) > 0\ \ s.\ t.\ |x - x_o|\ < \delta \Rightarrow\ |f_{n_o}(x) - f_{n_o}(x_o)| < \frac{\varepsilon}{3}$ --------- ------(3)

(1),(2) & (3) , *we have*

$\forall\ \ \varepsilon > 0\ \ \exists\delta_{n_o}(\varepsilon) > 0\ \ s.\ t.\ |x - x_o|\ < \delta$

$$
\begin{aligned}
\Rightarrow |f(x) - f(x_o)| &= |f(x) - f_{n_o}(x) + f_{n_o}(x) - f_{n_o}(x_o) + f_{n_o}(x_o) - f(x_o)| \\
&\prec\ |f(x) - f_{n_o}(x)| + |f_{n_o}(x) - f_{n_o}(x_o)| + |f_{n_o}(x_o) - f(x_o)| \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

Hence, f(x) is continous at any point of the domain D.

2) *upper Darbaux sum, lower Darboux sum,*

Let $f(x)$ *be a bounded real valued function on [a,b]. Under any partition* $P = \{a = x_o < x_1 < \ldots < x_n = b\}$ *of an interval [a,b], we define:*

$M_i := \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$

$m_i := \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$

$$
\Rightarrow \begin{cases} U(f.P) := \sum_{i=1}^{n} M_i \Delta x_i \\ L(f.P) = \sum_{i=1}^{n} m_i \Delta x_i \end{cases}
$$

U(f,p):upper Darbaux sum

L(f,P): lower Darbaux sum

Riemann integrable function.

Let $f(x)$ be a bounded real valued function on [a,b]. A function $f(x)$ is called Riemann intagrable if

$$\inf_P U(f.P) = \sup_P L(f.P)$$

$$\Rightarrow R\int_a^b f(x)dx = \inf_P U(f.P)$$

3) *Without loss of generality we may consider that the partition P^*, finer than the partition P, has exactly one point x^* more than P.*

Let $x_{i-1} < x^* < x_i$

- $U(f.P^*) \le U(f.P)$?

Let $M_i^* := \sup\{f(x) : x_{i-1} \le x \le x^*\}$

$M_i^{**} := \sup\{f(x) : x^* \le x \le x_i\}$

$\Rightarrow M_i^* \le M_i$ & $M_i^{**} \le M_i$

$\Rightarrow U(f.P) - U(f.P^*) = M_i(x_i - x_{i-1}) - M_i^*(x^* - x_{i-1}) - M_i^{**}(x_i - x^*)$

$= (M_i\text{-}M_i^*)(x^* - x_{i-1}) + (M_i\text{-}M_i^{**})(x_i - x^*) \ge 0$

$\Rightarrow U(f.P) \ge U(f.P^*)$ --------------------------------(1)

- Similarly, we can verify that

$L(f.P) \le L(f.P^*)$ --------------------------------(2)

(1) & (2) $\Rightarrow L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P)$.

4) *The sequence $\{f_n(x)\} = \left\{\dfrac{1}{1+nx}\right\}$ of functions on the interval [0,1] is pint-wise convergence sequence.*

$$\bullet \; For\ x=0 \;\Rightarrow \{f_n(x)\} = \left\{\frac{1}{1+nx}\right\} = \{1\} \qquad ------------(1)$$

$$\bullet \; For\ 0 < x \le 1 \;\Rightarrow \lim_{n\to\infty}\{f_n(x)\} = \lim_{n\to\infty}\left\{\frac{1}{1+nx}\right\} = 0 \quad ------(2)$$

$$(1)\&(2) \Rightarrow \lim_{n\to\infty}\{f_n(x)\} = f(x) := \begin{cases} 1 & for\ x=0 \\ 0 & for\ 0 < x \le 1 \end{cases}$$

$\Rightarrow \{f_n(x)\}$ converges pointwise to $f(x)$

• $f(x)$ is not continuous and by theorem

$\Rightarrow$ the convergence of $\{f_n(x)\}$ is not uniform convergence to $f(x)$ ****

انتهت الإجابة

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